

CHARACTERIZATIONS OF HARDY SPACES ON TUBE DOMAINS OVER POLYHEDRAL CONES

ZUNWEI FU, LOUKAS GRAFAKOS, WEI WANG AND QINGYAN WU

ABSTRACT. This paper is devoted to the equivalence of various characterizations of holomorphic H^1 Hardy spaces on tube domains over polyhedral cones. We establish a new iterated Poisson integral formula which reproduces holomorphic functions on such domains. However, this formula shows that holomorphic H^1 functions have boundary values in a new type of Hardy space of real variables on their Shilov boundaries $\mathbb{R}^n$, which cannot be treated by standard multi-parameter harmonic analysis. We overcome this difficulty by developing techniques suitably adapted to this setting. Using the iterated Poisson integral as our approximation to the identity, and employing a lifting technique, we introduce various notions of multi-parameter analysis adapted to tube domains, such as twisted rectangles, new non-tangential approach regions, non-tangential maximal functions and Littlewood-Paley type functions. All these notions exhibit new geometric features associated with polyhedral cones and involve hidden parameters, as in the flag setting. We develop the necessary multi-parameter tools to investigate these new Hardy spaces. In particular, we apply these tools to obtain equivalent characterizations of the holomorphic H^1 Hardy spaces on tube domains in terms of non-tangential maximal, Lusin-Littlewood-Paley area and Littlewood-Paley g -functions.

CONTENTS

1. Introduction	2
2. Preliminaries	8
2.1. Partial convolutions along lines and partial Laplace equations	8
2.2. A formula of integration by parts	10
3. New maximal functions associated to twisted rectangles and iterated Poisson integral formula	10
3.1. Twisted rectangles, iterated and twisted maximal functions	10
3.2. Boundary growth estimates for iterated Poisson integrals and holomorphic H^p -functions	13
3.3. Proof of iterated Poisson integral formula and harmonic control	18
4. The maximal function characterization	21
4.1. Reduction of the Fefferman-Stein type good- λ inequality to an integral estimate	21

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Zunwei Fu and Qingyan Wu: Department of Mathematics, Linyi University, Shandong 276005, China, Email: fuzunwei@lyu.edu.cn, wuqingyan@lyu.edu.cn;

Loukas Grafakos: Department of Mathematics, University of Missouri, Columbia, MO 65211, USA, Email: grafakosl@missouri.edu;

Wei Wang: Department of Mathematics, Zhejiang University, Zhejiang 310058, China, Email: wwang@zju.edu.cn.

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4.2. Estimates of integrals by differential identities	23
4.3. Estimates in the general case	25
4.4. Proof of the Fefferman-Stein type good- λ inequality	27
5. $\mathfrak{H}$ -valued Littlewood-Paley g function and S function	31
5.1. The Plancherel-Pólya type inequality for $\mathfrak{H}$ -valued functions	31
5.2. The estimate $\ f\ _{L^1} \lesssim \ g(f)\ _{L^1}$	34
6. Proof of the equivalence of various characterizations	37
Appendix A. The Plancherel-Pólya type inequality for $\mathfrak{H}$ -valued functions	40
References	46

1. INTRODUCTION

It is well known that holomorphic functions in the Hardy space $H^p(\mathbb{R}_+^2)$ on the upper half-plane $\mathbb{R}_+^2$ can be characterized by their non-tangential maximal functions or Lusin-Littlewood-Paley area functions, and that the real parts of the boundary values of such functions constitute the Hardy space $H^p(\mathbb{R})$ of real-variable functions on $\mathbb{R}$. In order to define the Hardy space $H^p(\mathbb{R}^n)$ of real variables on $\mathbb{R}^n$, Stein and Weiss [45] introduced the notion of a generalized analytic function $u = (u_0, \dots, u_n) : \mathbb{R}_+^{n+1} \rightarrow \mathbb{R}^{n+1}$, which satisfies

$$\sum_{j=0}^n \frac{\partial u_j}{\partial x_j} = 0, \quad \frac{\partial u_j}{\partial x_k} = \frac{\partial u_k}{\partial x_j} \quad \text{for } j \neq k,$$

and proved that the Hardy space $H^p(\mathbb{R}_+^{n+1})$ of generalized analytic functions for $p \geq \frac{n-1}{n}$ can also be characterized by non-tangential maximal functions or area functions. The components u_0 of the boundary values of such functions constitute the Hardy space $H^p(\mathbb{R}^n)$ of real variables on $\mathbb{R}^n$.

There is a long history of extending the classical characterizations of analytic Hardy spaces to more general pseudoconvex domains in several complex variables and of establishing a correspondence between the boundary values of analytic Hardy functions and suitable Hardy spaces of real variables on their Shilov boundaries. For the Siegel upper half-space, Geller [15] proved that these characterizations are equivalent and that the boundary values of holomorphic H^p functions belong to the Hardy space H^p on the Heisenberg group. Such characterizations can be extended to a large class of smooth pseudoconvex domains with smooth boundary (cf. e.g. [29, 41]).

However, for pseudoconvex domains with non-smooth boundary, new phenomena appear. Even for the bidisc, there are fundamental difficulties caused by the higher-dimensional geometric complexity. In the 1970s, Malliavin and Malliavin [33] proved that the maximal function characterization implies the characterization in terms of the Lusin-Littlewood-Paley area function, by introducing a technique based on an auxiliary function. This result was generalized to the H^p theory for the bidisc by Gundy and Stein [18]. Biharmonic or holomorphic H^p functions on the bidisc have boundary values in the bi-parameter Hardy space on its Shilov boundary, namely a torus. These results stimulated the development of multi-parameter harmonic analysis (cf. e.g. [4, 10, 26, 30, 38]), which remains an active field of research. Later, Sato [39] extended the theory to the equivalence of characterizations of Hardy spaces for generalized analytic functions on the product $\mathbb{R}_+^{n_1+1} \times \mathbb{R}_+^{n_2+1}$.

Since general Siegel domains or symmetric spaces have non-smooth boundaries, admissible convergence of Poisson integrals becomes very complicated (cf. e.g. [28, 37, 40, 43]), and the study of holomorphic H^p Hardy spaces is usually restricted to homogeneous Siegel domains and to $p > 1$ (cf. e.g. [1, 2] and the

references therein). However, in [47], the third and fourth named authors proved that any holomorphic H^1 function on the product of two Siegel upper half-spaces has a boundary value in a bi-parameter Hardy space H^1 on its Shilov boundary, which is the product of two Heisenberg groups. By using bi-parameter harmonic analysis (cf. e.g. [6]), the Cauchy-Szegő projection is used to decompose a holomorphic H^1 function into a sum of holomorphic atoms. In [48], the third and fourth named authors introduced and investigated a new class of flag-like singular integral kernels, modelled on the Cauchy-Szegő kernels on certain Siegel domains. These kernels are neither of product type nor of flag type on their Shilov boundaries, which have the structure of nilpotent Lie groups of step two, say $\mathcal{N}$. New Hardy spaces introduced in [48] are defined by iterated heat kernel integrals on these groups $\mathcal{N}$.

In general, holomorphic H^p functions on different domains may have boundary values in different types of Hardy spaces of real variables on their Shilov boundaries, and one needs to use different kinds of multi-parameter harmonic analysis to deal with them. Stein wrote in [44] that “a suitable version of multi-parameter analysis will provide the missing theory of singular integrals needed for a variety of questions in several complex variables. This is indeed an exciting prospect.” Under this philosophy, Nagel, Ricci and Stein [36] studied an important class of Siegel domains: tube domains over polyhedral cones. A *polyhedral cone* Ω in $\mathbb{R}^n$ is the interior of the convex hull of a finite number of rays meeting at the origin, among which there exist at least n that are linearly independent. The *tube domain* over Ω is the domain in $\mathbb{C}^n$ defined by

$$T_\Omega := \mathbb{R}^n + \mathbf{i}\Omega = \{x + \mathbf{i}y \in \mathbb{C}^n : x \in \mathbb{R}^n, y \in \Omega\},$$

whose Shilov boundary is the Euclidean space $\mathbb{R}^n + \mathbf{i}0$. A tube domain over a cone is a Siegel domain of the first kind. Following the standard convention for such domains, we assume throughout this paper that the polyhedral cone Ω is regular; that is, its dual cone $\Omega^* := \{\xi \in \mathbb{R}^n : y \cdot \xi \geq 0 \text{ for all } y \in \Omega\}$ has nonempty interior. This excludes degenerate cases such as $\Omega = \mathbb{R}^n$. The holomorphic Hardy space $H^p(T_\Omega)$ consists of all holomorphic functions F on T_Ω such that

$$(1.1) \quad \|F\|_{H^p(T_\Omega)} := \left(\sup_{y \in \Omega} \int_{\mathbb{R}^n} |F(x + \mathbf{i}y)|^p dx \right)^{1/p} < \infty.$$

See [46, Chapter 3] for basic facts about H^p Hardy spaces on tube domains. The Cauchy-Szegő projection $L^2(\mathbb{R}^n) \rightarrow H^2(T_\Omega)$ has a kernel, called the *Cauchy-Szegő kernel*, given by

$$C(x + \mathbf{i}y) = \int_{\Omega^*} e^{2\pi \mathbf{i}(x + \mathbf{i}y) \cdot \xi} d\xi.$$

Nagel, Ricci and Stein [36] proved that this kernel is a sum of flag kernels when restricted to the Shilov boundary. However, the study of the Hardy spaces associated to sums of flag kernels is still in its early stage (cf. e.g. [24] and the references therein). This paper is devoted to developing the necessary tools of a “suitable version of multi-parameter analysis” to investigate new types of Hardy spaces of real variables on the Shilov boundary $\mathbb{R}^n$, to applying them to the holomorphic H^1 Hardy spaces on T_Ω , and to proving the equivalence of several characterizations of this space.

It is well known [13] that Hardy spaces of real variables are defined in terms of approximations to the identity. Thus our first step is to find an integral representation formula for holomorphic functions on T_Ω , which can also be viewed as an approximation to the identity on the Shilov boundary $\mathbb{R}^n$. We use this representation to define new Hardy spaces corresponding to holomorphic H^1 Hardy spaces on T_Ω , and study the latter. For simplicity, we restrict ourselves to $H^p(T_\Omega)$ with $p = 1$ in this paper, and assume

that the polyhedral cone is spanned by unit vectors e_j in $\mathbb{R}^n$, $j = 1, \dots, m$, i.e.

$$\Omega = \{\lambda_1 e_1 + \dots + \lambda_m e_m : \lambda_1, \dots, \lambda_m > 0\},$$

and that every n of $e_1, \dots, e_m$ are linearly independent, to avoid degeneracy (see Figure 1). We must have $m \geq n$. $\mathcal{E} := \{e_1, \dots, e_m\}$ is said to be a *generating system* of the polyhedral cone Ω . By definition, Ω has the form $\{\lambda y : \lambda > 0, y \in \mathcal{C}\}$, where $\mathcal{C}$ is the convex hull of the vectors $\{e_1, \dots, e_m\}$, i.e. $\mathcal{C} = \{\lambda_1 e_1 + \dots + \lambda_m e_m : \lambda_1 + \dots + \lambda_m = 1, \lambda_1, \dots, \lambda_m > 0\}$.

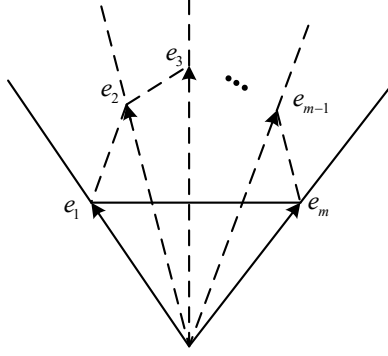


FIGURE 1. Polyhedral cone

Let $P_t(s) = \frac{1}{\pi} \frac{t}{t^2 + s^2}$ be the standard Poisson kernel on $\mathbb{R}$. Define the *Poisson integral along the line e_μ* by

$$(1.2) \quad f *_\mu P_t(x) := \int_{\mathbb{R}} f(x - se_\mu) P_t(s) ds$$

for $x \in \mathbb{R}^n$, and the *iterated Poisson integral* by

$$(1.3) \quad P_{\mathbf{t}}(f)(x) := f *_{\mathbf{t}} P_{t_1} \cdots *_{\mathbf{t}} P_{t_m}(x),$$

which is a function of the variables $x \in \mathbb{R}^n$ and $\mathbf{t} = (t_1, \dots, t_m) \in (\mathbb{R}_+)^m$. We often write $P_{\mathbf{t}}(f)(x)$ as $f * P_{\mathbf{t}}(x)$. As in the flag or flag-like cases (cf. e.g. [35, 48]), we consider the lifting space $\mathbb{R}^m$ and the projection $\pi : \mathbb{R}^m \rightarrow \mathbb{R}^n$ given by

$$(1.4) \quad (\lambda_1, \dots, \lambda_m) \mapsto \lambda_1 e_1 + \dots + \lambda_m e_m.$$

It follows from the classical boundary convergence theorem for Hardy spaces on tube domains (see [46, Chapter 3, Theorem 5.5]) that, if $F \in H^p(T_\Omega)$, then F has a boundary value

$$F^b := \lim_{y \in \Omega, y \rightarrow 0} F(\cdot + \mathbf{i}y) \in L^p(\mathbb{R}^n),$$

where the convergence is understood in $L^p(\mathbb{R}^n)$. More details will be recalled in Section 3. The following result gives an iterated Poisson integral representation of F in terms of this boundary value.

Theorem 1.1. *Suppose that T_Ω is a tube domain over a polyhedral cone as above. Let $F \in H^1(T_\Omega)$, and let F^b be its boundary value. Then*

$$(1.5) \quad F(x + \mathbf{i}\pi(\mathbf{t})) = F^b * P_{\mathbf{t}}(x),$$

for all $(x, \mathbf{t}) \in \mathbb{R}^n \times (\mathbb{R}_+)^m$.

Remark 1.1. (1) *This reproducing formula is unusual in the sense that we have to use the lifting cone $(\mathbb{R}_+)^m$, where a point y in the cone Ω is represented by $\pi(\mathbf{t})$. The iterated Poisson integral has an m -parameter structure, although $\mathbb{R}^n$ is only n -dimensional. Namely, there exist $m - n$ hidden parameters.*

(2) *This formula characterizes the new type of Hardy spaces of real variables on $\mathbb{R}^n$ to which the boundary values of holomorphic Hardy functions belong, and indicates what kind of multi-parameter analysis is needed to study them. We will develop the necessary tools of the corresponding multi-parameter harmonic analysis on the Shilov boundary $\mathbb{R}^n$ to solve problems concerning holomorphic Hardy functions on the tube domain.*

The iterated Poisson integral is our approximation to the identity; it allows us to introduce natural notions of multi-parameter analysis adapted to tube domains T_Ω , such as twisted rectangles, new non-tangential regions, non-tangential maximal functions, Lusin-Littlewood-Paley area functions and new types of Hardy spaces. All of these differ from the standard ones and have new geometric features associated with polyhedral cones.

For $\mathbf{t} := (t_1, \dots, t_m) \in (\mathbb{R}_+)^m$, the standard rectangle in $\mathbb{R}^m$ is $\tilde{R}(0, \mathbf{t}) = (-t_1, t_1) \times \dots \times (-t_m, t_m)$. Its image $\pi(\tilde{R}(0, \mathbf{t}))$ under the projection π in (1.4) is denoted by $R(0, \mathbf{t})$, and is called a *twisted rectangle* in $\mathbb{R}^n$. For $x \in \mathbb{R}^n$, let $R(x, \mathbf{t}) := x + R(0, \mathbf{t})$. Then

$$(1.6) \quad R(x, \mathbf{t}) := \{x + \lambda_1 e_1 + \dots + \lambda_m e_m : |\lambda_1| < t_1, \dots, |\lambda_m| < t_m\}.$$

For $x \in \mathbb{R}^n$ and any $\beta > 0$, we define the *non-tangential region* in $\mathbb{R}^n \times (\mathbb{R}_+)^m$ by

$$(1.7) \quad \Gamma_\beta(x) := \{(x', \mathbf{t}) \in \mathbb{R}^n \times (\mathbb{R}_+)^m : x' \in R(x, \beta \mathbf{t})\}.$$

We also denote $\Gamma_1(x)$ by $\Gamma(x)$ for short. Define the *non-tangential maximal function* on $\mathbb{R}^n$ by

$$(1.8) \quad N_\mathcal{E}^\beta(f)(x) := \sup_{(x', \mathbf{t}) \in \Gamma_\beta(x)} |(f * P_\mathbf{t})(x')|.$$

The Hardy space $H_{\max; \mathcal{E}}^1(\mathbb{R}^n)$ consists of $f \in L^1(\mathbb{R}^n)$ such that $N_\mathcal{E}^\beta(f) \in L^1(\mathbb{R}^n)$. These Hardy spaces for different $\beta > 0$ coincide with equivalent norms (cf. Lemma 3.1). For generating systems $\mathcal{E}$ and $\mathcal{E}'$ satisfying a mild assumption in Proposition 3.7, we must have $H_{\max; \mathcal{E}'}^1(\mathbb{R}^n) \neq H_{\max; \mathcal{E}}^1(\mathbb{R}^n)$. In particular, even two different generating systems for the same cone may give rise to different non-tangential regions, iterated Poisson structures, and hence different real-variable Hardy spaces. For example, the systems $\{(0, 1), (1, 0)\}$ and $\{(0, 1), (1, 0), (\sqrt{2}/2, \sqrt{2}/2)\}$ give rise to different real-variable Hardy spaces, even though both define the same cone $\mathbb{R}_+^2$.

Define the *Lusin-Littlewood-Paley area function* of f by

$$(1.9) \quad S_\mathcal{E}(f)(x) := \left(\int_{\Gamma(x)} |\nabla_1 \cdots \nabla_m (f * P_\mathbf{t})(x')|^2 \frac{\mathbf{t} \, d\mathbf{t} \, dx'}{|R(x, \mathbf{t})|} \right)^{1/2},$$

for all $x \in \mathbb{R}^n$, where ∇_j is the gradient in the plane spanned by e_j and t_j , and $\mathbf{t} \, d\mathbf{t} = t_1 \cdots t_m \, dt_1 \cdots dt_m$. Define the *S-function Hardy space* $H_{S; \mathcal{E}}^1(\mathbb{R}^n)$ by setting

$$H_{S; \mathcal{E}}^1(\mathbb{R}^n) := \{f \in L^1(\mathbb{R}^n) : S_\mathcal{E}(f) \in L^1(\mathbb{R}^n)\}$$

and equip it with the norm $\|f\|_{H_{S; \mathcal{E}}^1(\mathbb{R}^n)} := \|S_\mathcal{E}(f)\|_{L^1(\mathbb{R}^n)}$. The *Littlewood-Paley g-function* of f is defined by

$$(1.10) \quad g_\mathcal{E}(f)(x) := \left(\int_{(\mathbb{R}_+)^m} |\nabla_1 \cdots \nabla_m (f * P_\mathbf{t})(x)|^2 \mathbf{t} \, d\mathbf{t} \right)^{1/2},$$

for all $x \in \mathbb{R}^n$. We define the *g-function Hardy space*

$$H_{g,\mathcal{E}}^1(\mathbb{R}^n) := \{f \in L^1(\mathbb{R}^n) : g_{\mathcal{E}}(f) \in L^1(\mathbb{R}^n)\},$$

and this space is naturally equipped with the norm $\|f\|_{H_{g,\mathcal{E}}^1(\mathbb{R}^n)} := \|g_{\mathcal{E}}(f)\|_{L^1(\mathbb{R}^n)}$. If there is no ambiguity, we will omit the dependence of the notation on $\mathcal{E}$.

Note that the tube domain $T_{\Omega} = \mathbb{R}^n + \mathbf{i}\Omega$ is different from $\mathbb{R}^n \times (\mathbb{R}_+)^m$, but $\Omega = \pi((\mathbb{R}_+)^m)$. If we lift the function F on T_{Ω} to a function $u(x, \mathbf{t}) = F(x + \mathbf{i}\pi(\mathbf{t}))$ on $\mathbb{R}^n \times (\mathbb{R}_+)^m$, then by Theorem 1.1, $u(x, \mathbf{t}) = F^b * P_{\mathbf{t}}(x)$ if $F \in H^1(T_{\Omega})$. Motivated by nontangential maximal function, area function and *g-function* for iterated Poisson integrals above, we define the *nontangential maximal function* of a holomorphic function F on T_{Ω} by

$$(1.11) \quad \mathbb{N}_{\Omega}^{\beta}(F)(x) := \sup_{(x', \mathbf{t}) \in \Gamma_{\beta}(x)} |F(x' + \mathbf{i}\pi(\mathbf{t}))|,$$

for all $x \in \mathbb{R}^n$. We also define the *Lusin-Littlewood-Paley area function* by

$$(1.12) \quad \mathbb{S}_{\Omega}(F)(x) := \left(\int_{\Gamma(x)} |\nabla_1 \cdots \nabla_m (F(x' + \mathbf{i}\pi(\mathbf{t})))|^2 \frac{\mathbf{t} \, d\mathbf{t} \, dx'}{|R(x, \mathbf{t})|} \right)^{1/2},$$

and the *Littlewood-Paley g-function* by

$$\mathbb{G}_{\Omega}(F)(x) := \left(\int_{(\mathbb{R}_+)^m} |\nabla_1 \cdots \nabla_m (F(x + \mathbf{i}\pi(\mathbf{t})))|^2 \mathbf{t} \, d\mathbf{t} \right)^{1/2}.$$

Having established this notation, we are now able to state our main theorem.

Theorem 1.2. *Suppose that T_{Ω} is a tube domain over a polyhedral cone as above. Then, for a holomorphic function F on T_{Ω} , the following properties are equivalent:*

- (1) $F \in H^1(T_{\Omega})$;
- (2) For fixed $\beta > 0$, $\mathbb{N}_{\Omega}^{\beta}(F) \in L^1(\mathbb{R}^n)$;
- (3) $\lim_{\substack{|y| \rightarrow +\infty \\ y \in y_0 + \Omega}} F(x + \mathbf{i}y) = 0$ for every fixed $y_0 \in \Omega$, and $\mathbb{S}_{\Omega}(F) \in L^1(\mathbb{R}^n)$.
- (4) $\lim_{\substack{|y| \rightarrow +\infty \\ y \in y_0 + \Omega}} F(x + \mathbf{i}y) = 0$ for every fixed $y_0 \in \Omega$, and $\mathbb{G}_{\Omega}(F) \in L^1(\mathbb{R}^n)$.

Moreover,

$$\|F\|_{H^1(T_{\Omega})} \approx \|\mathbb{N}_{\Omega}^{\beta}(F)\|_{L^1(\mathbb{R}^n)} \approx \|\mathbb{S}_{\Omega}(F)\|_{L^1(\mathbb{R}^n)} \approx \|\mathbb{G}_{\Omega}(F)\|_{L^1(\mathbb{R}^n)}.$$

By Lemma 3.1, the condition (2) for fixed $\beta > 0$ is equivalent to the same condition for every $\beta > 0$.

Corollary 1.1. *Suppose that T_{Ω} is a tube domain over a polyhedral cone as above. Then, for $F \in H^1(T_{\Omega})$, its boundary limit $F^b := \lim_{y \in \Omega, y \rightarrow 0} F(x + \mathbf{i}y)$ belongs to $H_{\diamond, \mathcal{E}}^1(\mathbb{R}^n)$, where $\diamond \in \{\max, S, g\}$ and $\mathcal{E}$ is a generating system of the polyhedral cone Ω .*

Remark 1.2. *By Theorem 1.1, a holomorphic H^1 function is the iterated Poisson integral of its boundary L^1 limit on the Shilov boundary. Then, by employing the tools of multi-parameter analysis we have outlined and using the holomorphicity of F , we obtain a complete real variable characterization of holomorphic H^1 Hardy spaces on such tube domains. In fact we show*

$$(1.13) \quad H_{\max, \mathcal{E}}^1(\mathbb{R}^n) \subset H_{S, \mathcal{E}}^1(\mathbb{R}^n) \subset H_{g, \mathcal{E}}^1(\mathbb{R}^n)$$

to prove (2) $\Rightarrow$ (3) $\Rightarrow$ (4), where $\mathcal{E}$ is the generating system of the polyhedral cone Ω . It would be interesting to prove the equivalence of Hardy spaces in (1.13) only by real-variable methods. One possible way would be to develop a new kind of atomic decomposition and Journé's covering lemma [26, 38] adapted

to twisted rectangles. For the equivalence of characterizations of Hardy spaces of real variables in various settings, see, for example, [3, 5, 6, 9, 16, 21, 23, 49].

The proof of Theorem 1.2 proceeds as follows. For holomorphic functions F , we can use the subharmonicity of $|F|^p$ for $p > 0$ to prove (1) $\Rightarrow$ (2). The implication (2) $\Rightarrow$ (3) can be reduced to a Fefferman–Stein type good- λ inequality: there exist $C > 0$ and $\beta > 1$ such that for $f \in L^1(\mathbb{R}^n)$ and all $\lambda > 0$,

$$(1.14) \quad |\{x \in \mathbb{R}^n : S(f)(x) > \lambda\}| \leq C |\{x \in \mathbb{R}^n : N^\beta(f)(x) > \lambda\}| + \frac{C}{\lambda^2} \int_{\{x \in \mathbb{R}^n : N^\beta(f)(x) \leq \lambda\}} N^\beta(f)(x)^2 dx,$$

where C is a constant independent of f and λ . In the one-parameter case, this inequality was proved by Fefferman and Stein [11, 42] via smooth surface approximation and an application of Green’s formula in smooth domains. The main difficulty in the bi-parameter case is the lack of such an effective approximation, owing to the higher-dimensional geometric complexity. To overcome this difficulty for biharmonic functions, Malliavin and Malliavin [33] introduced an auxiliary function. So far, maximal function characterizations have been established only in a few cases by proving analogues of (1.14): the $\mathbb{R}^n \times \mathbb{R}^n$ case by Merryfield [34], the Muckenhoupt–Stein Bessel operator setting by Duong, Li, Wick, Yang [9], the multi-parameter flag setting by Han, Lee, Li, Wick [21, 22], product stratified Lie groups by Cowling, Fan, Li, Yan [7], the flag setting on the Heisenberg group by Chen, Cowling, Lee, Li, Ottazzi [5] and the Shilov boundary of products of domains of finite type by Li [32], etc. In our setting, the difficulty lies in the fact that the number of parameters is arbitrary, unlike the aforementioned situations which involve only two. Thus we have to find general inductive formulae to overcome this difficulty.

In the product case $\mathbb{R}_+^{n_1+1} \times \mathbb{R}_+^{n_2+1}$, the implication (3) $\Rightarrow$ (1) was proved by use of the estimate $\|\mathfrak{f}\|_1 \lesssim \|S(\mathfrak{f})\|_1$ for Hilbert space-valued functions $\mathfrak{f}$ of one variable [39]. Because our non-tangential region $\Gamma_\beta(x)$ in (1.7) cannot be written as a product, we cannot employ this approach. However, since the domain $(\mathbb{R}_+)^m$ of integration for the g -function has a product structure, we add condition (4), which usually does not appear in characterizations of (generalized) analytic Hardy spaces (cf. e.g. [15, 39, 45]). The implication (3) $\Rightarrow$ (4) follows from a pointwise control of the $\mathbb{G}$ -function by the $\mathbb{S}$ -function via the mean value formula. The real-variable version of (4) $\Rightarrow$ (1) is

$$(1.15) \quad \sup_{\mathbf{t} \in (\mathbb{R}_+)^m} \|f * P_{\mathbf{t}}\|_{L^1(\mathbb{R}^n)} \lesssim \|g(f)\|_{L^1(\mathbb{R}^n)}.$$

We first establish a one-dimensional version of (1.15) for Hilbert space-valued functions by applying the corresponding one-dimensional Plancherel–Pólya type inequality. Then we use this estimate iteratively to prove (1.15).

The paper is organized as follows. In Section 2, we discuss partial convolutions with the Poisson kernel along lines, which satisfy partial Laplace equations. In Section 3, we characterize the geometric shape of twisted rectangles and the associated twisted maximal functions. We also obtain boundary growth estimates for iterated Poisson integrals and for holomorphic H^p -functions, respectively, and then show that the iterated Poisson integral reproduces holomorphic functions by using Fourier transform. In Section 4, the Fefferman–Stein type good- λ inequality (1.14) is reduced to the estimate of an integral over $\mathbb{R}^n \times (\mathbb{R}_+)^m$ by means of an auxiliary function. We use differential identities to estimate the terms appearing in this integral, the new terms arising after these estimates, and in general all terms appearing inductively. In Section 5, we prove a one-dimensional version of (1.15) for Hilbert space-valued functions, from which we deduce the real-variable version (1.15) of (4) $\Rightarrow$ (1). In Section 6, we prove the equivalence of the various characterizations in the main Theorem 1.2. In the appendix, we give the

proof of the Plancherel-Pólya type inequality for Hilbert space-valued functions for the convenience of the readers.

2. PRELIMINARIES

2.1. Partial convolutions along lines and partial Laplace equations. Consider the *complex line* in $\mathbb{C}^n$ that contains e_μ :

$$(2.1) \quad L_\mu := \{(s + \mathbf{i}t)e_\mu : s, t \in \mathbb{R}\},$$

and its upper half plane

$$(2.2) \quad L_\mu^+ = \{(s + \mathbf{i}t)e_\mu : s \in \mathbb{R}, t > 0\},$$

with boundary $\mathbb{R}e_\mu$ contained in the Shilov boundary $\mathbb{R}^n + \mathbf{i}0$ of T_Ω . Then $x + L_\mu^+$ may be contained in the boundary of the tube domain for any $x \in \mathbb{R}^n$, because e_μ may be contained in $\partial\Omega$. But we always have $x + iy + L_\mu^+ \subset T_\Omega$ for $y \in \Omega$.

Denote by $*_\mu$ the *partial convolution on $\mathbb{R}^n$ along the line e_μ* , i.e. for $f \in L^1(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R})$,

$$(2.3) \quad f *_\mu g(x) := \int_{\mathbb{R}} f(x - se_\mu)g(s) ds,$$

for almost all $x \in \mathbb{R}^n$. It is immediate that partial convolutions along two lines e_μ and e_ν are commutative:

$$(2.4) \quad f *_\mu g *_\nu h(x) = f *_\nu h *_\mu g(x),$$

for any $g, h \in L^1(\mathbb{R})$. If we write

$$(2.5) \quad e_\mu = (e_{\mu 1}, \dots, e_{\mu n}) \in \mathbb{R}^n,$$

$\mu = 1, \dots, m$, then the vector field

$$(2.6) \quad X_\mu := \sum_{k=1}^n e_{\mu k} \frac{\partial}{\partial x_k},$$

is the derivative along the direction e_μ . Define the μ -th *Laplacian* and the gradient of a function u along the upper half plane L_μ^+ in (2.2) as

$$\Delta_\mu := X_\mu^2 + \frac{\partial^2}{\partial t_\mu^2}, \quad \nabla_\mu u := \left(X_\mu u, \frac{\partial u}{\partial t_\mu} \right),$$

respectively, $\mu = 1, \dots, m$. For fixed μ , we will use coordinates $x = x^\perp + se_\mu$ for $x \in \mathbb{R}^n$, where $x^\perp$ is the part of x perpendicular to e_μ . Let $e_\mu^\perp$ be the subspace perpendicular to e_μ .

Proposition 2.1. (1) *If F is holomorphic on the tube domain T_Ω , then for fixed $x + \mathbf{i}y \in T_\Omega$ and $\mu \in \{1, 2, \dots, m\}$, the restriction of F to $x + \mathbf{i}y + L_\mu^+$ is holomorphic, i.e. $F(x + \mathbf{i}y + ze_\mu)$ is holomorphic in $z \in \mathbb{C}$ for $\text{Im } z > 0$.*

(2) *If $f \in L^p(\mathbb{R}^n)$ ($1 \leq p < \infty$) and $\mu \in \{1, 2, \dots, m\}$, then for almost every $x^\perp$, $f *_\mu P_{t_\mu}(x^\perp + se_\mu)$ is smooth in s, t_μ and satisfies*

$$(2.7) \quad \Delta_\mu[f *_\mu P_{t_\mu}](x^\perp + se_\mu) = 0, \quad \nabla_\mu[f *_\mu P_{t_\mu}](x^\perp + se_\mu) = f *_\mu \widetilde{\nabla}_\mu P_{t_\mu}(x^\perp + se_\mu),$$

where $\widetilde{\nabla}_\mu$ is the gradient in the plane with variables s, t_μ : $\widetilde{\nabla}_\mu v = \left(\frac{\partial v}{\partial s}, \frac{\partial v}{\partial t_\mu} \right)$.

Proof. (1) If we write the coordinates $z_1, \dots, z_n$ of the tube domain as $z_j = x_j + \mathbf{i}y_j$ and $z = s + \mathbf{i}t$, then, for fixed μ we have

$$\begin{aligned} 2\frac{\partial F}{\partial \bar{z}} &= \left(\frac{\partial}{\partial s} + \mathbf{i} \frac{\partial}{\partial t} \right) [F(x + \mathbf{i}y + (s + \mathbf{i}t)e_\mu)] \\ &= \sum_{k=1}^n e_{\mu k} \frac{\partial F}{\partial x_k} (x + \mathbf{i}y + (s + \mathbf{i}t)e_\mu) + \mathbf{i} \sum_{k=1}^n e_{\mu k} \frac{\partial F}{\partial y_k} (x + \mathbf{i}y + (s + \mathbf{i}t)e_\mu) \\ &= \sum_{k=1}^n e_{\mu k} \left(\frac{\partial F}{\partial x_k} + \mathbf{i} \frac{\partial F}{\partial y_k} \right) (x + \mathbf{i}y + (s + \mathbf{i}t)e_\mu) = 0, \end{aligned}$$

by using (2.5) and the Cauchy-Riemann equation satisfied by F .

(2) Since $f \in L^p(\mathbb{R}^n)$, we see that for almost all $x^\perp \in e_\mu^\perp$, $f_{x^\perp}(s) := f(x^\perp + se_\mu)$ belongs to $L^p(\mathbb{R})$. Thus, the function $(s, t) \mapsto (f *_\mu P_t)(x^\perp + se_\mu) = f_{x^\perp} * P_t(s)$ is harmonic (hence smooth) for almost every $x^\perp$. Note that

$$\begin{aligned} (2.8) \quad X_\mu(f *_\mu P_t)(x^\perp + se_\mu) &= \frac{\partial}{\partial s} [(f *_\mu P_t)(x^\perp + se_\mu)] = \frac{\partial}{\partial s} [f_{x^\perp} * P_t(s)] \\ &= \left[f_{x^\perp} * \frac{\partial}{\partial s} P_t \right] (s) = \left[f *_\mu \left(\frac{\partial}{\partial s} P_t \right) \right] (x^\perp + se_\mu), \end{aligned}$$

by definition (2.6), and a similar identity holds for $\frac{\partial}{\partial t_\mu}$. We get the second identity in (2.7), and so

$$\Delta_\mu [f *_\mu P_{t_\mu}(x)] = \left[f *_\mu \left(\frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial t_\mu^2} \right) P_{t_\mu} \right] (x^\perp + se_\mu) = 0.$$

The proposition is proved. $\square$

Write

$$(2.9) \quad \nabla_\mu := (X_\mu^{(1)}, X_\mu^{(2)}), \quad \text{where } X_\mu^{(1)} = X_\mu, \quad X_\mu^{(2)} = \partial_{t_\mu}.$$

Denote $\mathbf{m} := \{1, \dots, m\}$. For a subset $\mathbf{j} = \{j_1, \dots, j_a\}$ of $\mathbf{m}$, we introduce the notation

$$(2.10) \quad \mathbf{t}_j := (t_{j_1}, \dots, t_{j_a}), \quad \nabla_j v := \nabla_{j_1} \cdots \nabla_{j_a} v, \quad \mathbf{t}_j d\mathbf{t}_j := t_{j_1} \cdots t_{j_a} dt_{j_1} \cdots dt_{j_a}.$$

Then $\nabla_j v$ in (2.10) is a $\mathbb{R}^{2^a}$ -valued function, and the inner product of two such vectors is defined by

$$\nabla_j u \cdot \nabla_j v := \sum_{\alpha_1, \dots, \alpha_a=1,2} X_{j_1}^{(\alpha_1)} \cdots X_{j_a}^{(\alpha_a)} u \cdot X_{j_1}^{(\alpha_1)} \cdots X_{j_a}^{(\alpha_a)} v,$$

where $X_{j_\mu}^{(\alpha_\mu)}$'s are given by (2.9). For $f \in L^1(\mathbb{R}^n)$ and $\phi \in L^1(\mathbb{R})$, we set

$$f *_{\mathbf{j}} \phi_{\mathbf{t}_j} := f *_{j_1} \phi_{t_{j_1}} \cdots *_{j_a} \phi_{t_{j_a}}(x),$$

where $\phi_t(s) = \frac{1}{t} \phi\left(\frac{s}{t}\right)$. Let

$$(2.11) \quad \mathbf{m}_k = \{k, \dots, m\}.$$

In view of this notation, we have $\mathbf{t}_\mathbf{m} = \mathbf{t}$. We will write $f * \phi_\mathbf{t}$ instead of $f *_{\mathbf{j}} \phi_{\mathbf{t}_j}$ when $\mathbf{j} = \mathbf{m}$. For a subset $\mathbf{j}$ of $\mathbf{m}$ and $f \in L^p(\mathbb{R}^n)$ ($1 \leq p < \infty$), define the *partial Littlewood-Paley g-function* of f by

$$g_j(f)(x) := \left(\int_{(\mathbb{R}_+)^{|\mathbf{j}|}} |\nabla_j (f *_{\mathbf{j}} P_{\mathbf{t}_j})(x)|^2 \mathbf{t}_j d\mathbf{t}_j \right)^{1/2},$$

where $|\mathbf{j}|$ denotes the number of elements in the set $\mathbf{j}$.

Proposition 2.2. *For $f \in L^p(\mathbb{R}^n)$ with $1 < p < \infty$, we have $\|g_j(f)\|_p \lesssim \|f\|_p$.*

The proof is standard (cf. e.g. [23, 48]) by repeatedly using the L^p boundedness of g -function for vector-valued functions. See also the proof of Proposition 5.6 for the converse part when $p = 1$. We omit the details.

2.2. A formula of integration by parts. This formula will be repeatedly used in the proof of good- λ inequality. We denote $\nabla_\mu^2 U := \nabla_\mu \nabla_\mu U$ for short, which is defined in (2.10), and $\nabla_\mu^0 U := U$.

Proposition 2.3. *Suppose that $U \in \mathcal{C}^2(\overline{\mathbb{R}_+^{n+1}})$ and satisfies the following three properties:*

- (1) *For $a = 0, 1, 2$, we have $\nabla_\mu^a U \in L^1(\mathbb{R}^n \times [0, T])$ for any $T > 0$, and for fixed $t_\mu > 0$, we have $\nabla_\mu^a U(\cdot, t_\mu) \in L^1(\mathbb{R}^n)$, and $X_\mu U(x^\perp + se_\mu, t_\mu) \rightarrow 0$ as $s \rightarrow \pm\infty$ for almost all $x^\perp \in e_\mu^\perp$.*
- (2) *As $t_\mu \rightarrow +\infty$, we have $U(\cdot, t_\mu), t_\mu \partial_{t_\mu} U(\cdot, t_\mu) \rightarrow 0$ in $L^1(\mathbb{R}^n)$;*
- (3) *As $t_\mu \rightarrow 0$, we have $U(\cdot, t_\mu) \rightarrow U(\cdot, 0)$ and $t_\mu \partial_{t_\mu} U(\cdot, t_\mu) \rightarrow 0$ in $L^1(\mathbb{R}^n)$.*

Then

$$(2.12) \quad \int_0^\infty \int_{\mathbb{R}^n} \Delta_\mu U(x, t_\mu) t_\mu dx dt_\mu = \int_{\mathbb{R}^n} U(x, 0) dx,$$

where $\int_0^\infty$ is only a short-hand notation for $\lim_{\varepsilon \rightarrow 0^+, T \rightarrow +\infty} \int_\varepsilon^T$.

Proof. Write

$$\int_\varepsilon^T \int_{\mathbb{R}^n} \Delta_\mu U(x, t_\mu) t_\mu dx dt_\mu = \int_\varepsilon^T \int_{\mathbb{R}^n} X_\mu^2 U(x, t_\mu) t_\mu dx dt_\mu + \int_\varepsilon^T \int_{\mathbb{R}^n} \partial_{t_\mu}^2 U(x, t_\mu) t_\mu dx dt_\mu =: I_1 + I_2.$$

Note that $x \mapsto (x^\perp, s)$ preserves the volume form since we assume e_μ to be a unit vector. Hence,

$$\begin{aligned} \int_{\mathbb{R}^n} X_\mu^2 U(x, t_\mu) dx &= \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \frac{d^2}{ds^2} [U(x^\perp + se_\mu, t_\mu)] ds dx^\perp \\ &= \int_{\mathbb{R}^{n-1}} \left(\lim_{s \rightarrow +\infty} X_\mu U(x^\perp + se_\mu, t_\mu) - \lim_{s \rightarrow -\infty} X_\mu U(x^\perp + se_\mu, t_\mu) \right) dx^\perp = 0 \end{aligned}$$

by definition of X_μ and the assumption (1). Thus, $I_1 = 0$. For the term I_2 , we get

$$\begin{aligned} I_2 &= - \int_{\mathbb{R}^n} \int_\varepsilon^T \partial_{t_\mu} U(x, t_\mu) dt_\mu dx + \int_{\mathbb{R}^n} T \partial_{t_\mu} U(x, T) dx - \int_{\mathbb{R}^n} \varepsilon \partial_{t_\mu} U(x, \varepsilon) dx \\ &= - \int_{\mathbb{R}^n} U(x, T) dx + \int_{\mathbb{R}^n} U(x, \varepsilon) dx + \int_{\mathbb{R}^n} T \partial_{t_\mu} U(x, T) dx - \int_{\mathbb{R}^n} \varepsilon \partial_{t_\mu} U(x, \varepsilon) dx, \end{aligned}$$

by integration by parts. Then taking limit $T \rightarrow +\infty$ and $\varepsilon \rightarrow 0$, we deduce the required conclusions in view of assumptions (2) and (3). $\square$

3. NEW MAXIMAL FUNCTIONS ASSOCIATED TO TWISTED RECTANGLES AND ITERATED POISSON INTEGRAL FORMULA

3.1. Twisted rectangles, iterated and twisted maximal functions. Define the maximal function along the line e_μ as

$$M_\mu(f)(x) := \sup_{r_\mu \in \mathbb{R}_+} \frac{1}{2r_\mu} \int_{|s| < r_\mu} |f(x + se_\mu)| ds,$$

for $x \in \mathbb{R}^n$ and $f \in L^1_{loc}(\mathbb{R}^n)$. It is convenient to work with the iterated maximal function $M_{it} := M_m \circ \dots \circ M_1$. Then for $f \in L^1_{loc}(\mathbb{R}^n)$, there exists an absolute constant $C_0 > 0$ such that

$$(3.1) \quad |P_{\mathbf{t}}(f)(x')| \leq C_0 M_{it}(f)(x)$$

for any $(x', \mathbf{t}) \in \Gamma(x)$, by the property of the Poisson kernel [42]. Define the twisted maximal function as

$$M_{\mathbf{t}}(f)(x) := \sup_{\mathbf{r} \in (\mathbb{R}_+)^m} \frac{1}{|R(x, \mathbf{r})|} \int_{R(x, \mathbf{r})} |f(x')| dx'.$$

Note that the projection of the ball in the lifting space is usually called a tube in the flag or flag-like cases (cf. e.g. [5, 48]). But in this paper, the projection $R(x, \mathbf{r})$ in (1.6) is called twisted rectangle to avoid confusion with the tube domain. Given $\mathbf{r} = (r_1, \dots, r_m) \in (\mathbb{R}_+)^m$, let $r_{l_1}, \dots, r_{l_n}$ be the largest n numbers among $\{r_1, \dots, r_m\}$, i.e. $r_j \geq r_\mu$ for any $j \in \mathbf{l} := \{l_1, \dots, l_n\}$ and any $\mu \notin \mathbf{l}$. Set

$$(3.2) \quad R_{\mathbf{l}}(x, \mathbf{r}) := \{x + \lambda_{l_1} e_{l_1} + \dots + \lambda_{l_n} e_{l_n} : |\lambda_{l_1}| < r_{l_1}, \dots, |\lambda_{l_n}| < r_{l_n}\},$$

which is a parallelotope. Note that, given $\mathbf{r}$, the choice of $\mathbf{l}$ may not be unique.

On the other hand, given any subset $\mathbf{l} := \{l_1, \dots, l_n\}$ of $\mathbf{m}$, $e_{l_1}, \dots, e_{l_n}$ are linearly independent by the assumption. So there exist numbers $A_{\mu j}^{\mathbf{l}}$ such that $e_\mu = \sum_{j=1}^n A_{\mu j}^{\mathbf{l}} e_{l_j}$, for $\mu \notin \mathbf{l}$. As the number of choices of such subsets $\mathbf{l}$ is finite, we can define

$$(3.3) \quad \mathcal{A} := \sum_{\mathbf{l}} \sum_{\mu \notin \mathbf{l}} \sum_{j=1}^n |A_{\mu j}^{\mathbf{l}}| < +\infty.$$

Proposition 3.1. *Given $\mathbf{r} \in (\mathbb{R}_+)^m$, let $r_{l_1}, \dots, r_{l_n}$ be the largest n numbers among $\{r_1, \dots, r_m\}$. Then*

$$(3.4) \quad R_{\mathbf{l}}(x, \mathbf{r}) \subset R(x, \mathbf{r}) \subset R_{\mathbf{l}}(x, \tilde{\gamma}_0^{-1} \mathbf{r}),$$

for any $x \in \mathbb{R}^n$, where $\tilde{\gamma}_0 = (1 + \mathcal{A})^{-1} < 1$.

Proof. The first inclusion is obvious by comparing definitions (1.6) and (3.2). For the second one, note that for any $x' \in R(x, \mathbf{r})$, we can write

$$x' = x + \lambda_1 e_1 + \dots + \lambda_m e_m = x + \sum_{j=1}^n \left(\lambda_{l_j} + \sum_{\mu \notin \mathbf{l}} A_{\mu j}^{\mathbf{l}} \lambda_\mu \right) e_{l_j},$$

for some $|\lambda_1| < r_1, \dots, |\lambda_m| < r_m$. Then

$$\left| \lambda_{l_j} + \sum_{\mu \notin \mathbf{l}} A_{\mu j}^{\mathbf{l}} \lambda_\mu \right| < (1 + \mathcal{A}) r_{l_j} = \tilde{\gamma}_0^{-1} r_{l_j},$$

since $|\lambda_\mu| < r_{l_j}$ for $\mu \notin \mathbf{l}$ by definition. □

As a corollary, we get

$$(3.5) \quad |R(x, \mathbf{r})| \approx |R_{\mathbf{l}}(x, \mathbf{r})| = 2^n r_{l_1} \dots r_{l_n} |\det(e_{l_1}, \dots, e_{l_n})|.$$

Let χ be the characteristic function of the interval $[-1, 1]$ and let $\chi_t(s) := \frac{1}{t} \chi(\frac{s}{t})$ for $t \in \mathbb{R}^+$ and $s \in \mathbb{R}$. Recall that $|f| * \chi_{\mathbf{t}} := |f| *_{j_1} \chi_{t_{j_1}} \dots *_{j_m} \chi_{t_{j_m}}$ for $\mathbf{t} = (t_{j_1}, \dots, t_{j_m}) \in (\mathbb{R}_+)^m$. The twisted maximal function is controlled by iterated maximal function by the following proposition.

Proposition 3.2. *Suppose that $f \in L^1_{loc}(\mathbb{R}^n)$ and $m \geq n$. Then, for all $x \in \mathbb{R}^n$ and $\mathbf{t} \in (\mathbb{R}_+)^m$,*

$$|f| * \chi_{\gamma_0 \mathbf{t}}(x) \lesssim \frac{1}{|R(x, \mathbf{t})|} \int_{R(x, \mathbf{t})} |f(x')| dx' \lesssim |f| * \chi_{\gamma_0^{-1} \mathbf{t}}(x),$$

where $\gamma_0 = (1 + \mathcal{A})^{-2}$. Consequently, $M_{\mathbf{t}}(f) \lesssim M_{\text{it}}(f)$.

Proof. The case $m = n$ is immediate. Without loss of generality, we may assume that $t_1, \dots, t_n$ are the largest n numbers among $\{t_1, \dots, t_m\}$. The linear independence of $e_1, \dots, e_n$ implies that

$$(3.6) \quad e_\mu = \sum_{j=1}^n A_{\mu j} e_j,$$

for some numbers $A_{\mu j}$ with $\mu = n+1, \dots, m$. Then

$$(3.7) \quad \begin{aligned} |f| * \chi_{\mathbf{t}}(x) &= \int_{\mathbb{R}^m} \left| f \left(x - \sum_{j=1}^m s_j e_j \right) \right| \chi_{t_1}(s_1) \cdots \chi_{t_m}(s_m) ds_1 \cdots ds_m \\ &= \frac{1}{t_1 \cdots t_m} \int_{[-t_1, t_1] \times \cdots \times [-t_m, t_m]} \left| f \left(x - \sum_{j=1}^n \left(s_j + \sum_{\mu=n+1}^m A_{\mu j} s_\mu \right) e_j \right) \right| ds_1 \cdots ds_m. \end{aligned}$$

Take the coordinate transformation $\Theta : \mathbb{R}^m \rightarrow \mathbb{R}^m$, $\mathbf{s} \mapsto \mathbf{s}'$ given by

$$(3.8) \quad s'_j = \begin{cases} s_j + \sum_{\mu=n+1}^m A_{\mu j} s_\mu, & j = 1, \dots, n, \\ s_j, & j = n+1, \dots, m. \end{cases}$$

It is obvious that Θ is one to one and its Jacobian identically equals to 1. We claim that

$$(3.9) \quad \prod_{a=1}^m [-\tilde{\gamma}_0 t_a, \tilde{\gamma}_0 t_a] \subset \Theta \left(\prod_{a=1}^m [-t_a, t_a] \right) \subset \prod_{a=1}^m [-\tilde{\gamma}_0^{-1} t_a, \tilde{\gamma}_0^{-1} t_a],$$

with $\tilde{\gamma}_0 = (1 + \mathcal{A})^{-1}$. For the first inclusion, note that for each $\mathbf{s}' \in \prod_{a=1}^m [-\tilde{\gamma}_0 t_a, \tilde{\gamma}_0 t_a]$, there exists a unique preimage $\mathbf{s}$ in $\prod_{a=1}^m [-t_a, t_a]$, i.e. $\Theta(\mathbf{s}) = \mathbf{s}'$. This is because $|s_j| \leq t_j$ obviously for $j = n+1, \dots, m$ by definition (3.8), while for $j = 1, \dots, n$, we have

$$|s_j| = \left| s'_j - \sum_{\mu=n+1}^m A_{\mu j} s'_\mu \right| \leq \tilde{\gamma}_0 t_j + \mathcal{A} \tilde{\gamma}_0 t_j = t_j,$$

by $|s'_\mu| \leq \tilde{\gamma}_0 t_\mu \leq \tilde{\gamma}_0 t_j$ for any $\mu > n \geq j$ by the assumption. It is similar to show the second inclusion.

Now it follows from (3.7) and (3.9) that

$$\begin{aligned} |f| * \chi_{\mathbf{t}}(x) &\geq \frac{1}{t_1 \cdots t_m} \int_{[-\tilde{\gamma}_0 t_1, \tilde{\gamma}_0 t_1] \times \cdots \times [-\tilde{\gamma}_0 t_m, \tilde{\gamma}_0 t_m]} \left| f \left(x - \sum_{j=1}^n s'_j e_j \right) \right| ds'_1 \cdots ds'_m \\ &= \frac{(2\tilde{\gamma}_0)^{m-n}}{t_1 \cdots t_n} \int_{[-\tilde{\gamma}_0 t_1, \tilde{\gamma}_0 t_1] \times \cdots \times [-\tilde{\gamma}_0 t_n, \tilde{\gamma}_0 t_n]} \left| f \left(x - \sum_{j=1}^n s'_j e_j \right) \right| ds'_1 \cdots ds'_n \\ &= \frac{\tilde{\gamma}_0^{m-n} 2^m}{|R_{\mathbf{t}}(0, \tilde{\gamma}_0 \mathbf{t})|} \int_{R_{\mathbf{t}}(0, \tilde{\gamma}_0 \mathbf{t})} |f(x - x')| dx' \\ &\geq \frac{\tilde{\gamma}_0^{m-n} 2^m}{|R_{\mathbf{t}}(0, \mathbf{t})|} \int_{R(0, \tilde{\gamma}_0^2 \mathbf{t})} |f(x - x')| dx', \end{aligned}$$

by taking coordinates transformation (3.8) and using the transformation $\mathbb{R}^n \rightarrow \mathbb{R}^n$ given by

$$(3.10) \quad (s'_1, \dots, s'_n) \mapsto x' = \sum_{j=1}^n s'_j e_j$$

and Proposition 3.1. The Jacobian of this transformation is $\det(e_1, \dots, e_n) \neq 0$ since $e_1, \dots, e_n$ are linearly independent. Similarly,

$$\begin{aligned} |f * \chi_{\mathbf{t}}(x)| &\leq \frac{1}{t_1 \cdots t_m} \int_{[-\tilde{\gamma}_0^{-1}t_1, \tilde{\gamma}_0^{-1}t_1] \times \cdots \times [-\tilde{\gamma}_0^{-1}t_m, \tilde{\gamma}_0^{-1}t_m]} \left| f \left(x - \sum_{j=1}^n s'_j e_j \right) \right| ds'_1 \cdots ds'_m \\ &\approx \frac{1}{|R_{\mathbf{t}}(0, \mathbf{t})|} \int_{R_{\mathbf{t}}(0, \tilde{\gamma}_0^{-1}\mathbf{t})} |f(x - x')| dx'. \end{aligned}$$

The required conclusion follows by applying Proposition 3.1. Taking the supremum over $\mathbf{t} \in (\mathbb{R}_+)^m$ in the second inequality and using

$$|f| * \chi_{\tilde{\gamma}_0^{-1}\mathbf{t}}(x) \lesssim M_m \cdots M_1(f)(x) = M_{\text{it}}(f)(x),$$

we obtain $M_{\mathbf{t}}(f)(x) \lesssim M_{\text{it}}(f)(x)$. $\square$

Remark 3.1. *The control of the twisted maximal function by the iterated maximal function was already proved in the flag case [5] and the flag-like case [48, Proposition 3.2]. But the proof given here is simpler than that in [48].*

Lemma 3.1. *For $0 < p < \infty$ and $0 < \beta_1 \leq \beta_2$, $\|N_{\mathcal{E}}^{\beta_1}(f)\|_{L^p(\mathbb{R}^n)} \approx \|N_{\mathcal{E}}^{\beta_2}(f)\|_{L^p(\mathbb{R}^n)}$, i.e. the definition of $H_{\max; \mathcal{E}}^1(\mathbb{R}^n)$ is independent of β .*

Proof. Obviously, $N_{\mathcal{E}}^{\beta_1}(f) \leq N_{\mathcal{E}}^{\beta_2}(f)$. Let $(x', \mathbf{t}) \in \Gamma_{\beta_2}(x)$. For every $x'' \in R(x', \beta_1 \mathbf{t})$, the symmetry of the twisted rectangle implies that $x' \in R(x'', \beta_1 \mathbf{t})$, and hence $|f * P_{\mathbf{t}}(x')| \leq N_{\mathcal{E}}^{\beta_1}(f)(x'')$. Therefore,

$$|f * P_{\mathbf{t}}(x')|^q \leq \frac{1}{|R(x', \beta_1 \mathbf{t})|} \int_{R(x', \beta_1 \mathbf{t})} N_{\mathcal{E}}^{\beta_1}(f)(x'')^q dx'',$$

for $q > 0$. Since $x' \in R(x, \beta_2 \mathbf{t})$, we have $R(x', \beta_1 \mathbf{t}) \subset R(x, (\beta_1 + \beta_2) \mathbf{t})$ with equivalent volume. It follows that

$$|f * P_{\mathbf{t}}(x')|^q \lesssim_{\beta_1, \beta_2} M_{\mathbf{t}}(N_{\mathcal{E}}^{\beta_1}(f)^q)(x) \lesssim M_{\text{it}}(N_{\mathcal{E}}^{\beta_1}(f)^q)(x),$$

where the last inequality follows from Proposition 3.2. Taking the supremum over $(x', \mathbf{t}) \in \Gamma_{\beta_2}(x)$ gives

$$N_{\mathcal{E}}^{\beta_2}(f)(x)^q \lesssim M_{\text{it}}(N_{\mathcal{E}}^{\beta_1}(f)^q)(x).$$

Now we choose q such that $0 < q < p$. Then, the $L^{p/q}$ boundedness of M_{it} yields $\|N_{\mathcal{E}}^{\beta_2}(f)\|_{L^p}^q \lesssim \|N_{\mathcal{E}}^{\beta_1}(f)\|_{L^p}^q$. This proves the equivalence. $\square$

3.2. Boundary growth estimates for iterated Poisson integrals and holomorphic H^p -functions.

In the sequel, we will use the notation $\varepsilon := (\varepsilon, \dots, \varepsilon) \in (\mathbb{R}_+)^m$ for $\varepsilon > 0$, and the smoothing technique of replacing $f \in L^p(\mathbb{R}^n)$ by smooth function $f_{\varepsilon} = f * P_{\varepsilon}$ will be frequently used.

Proposition 3.3. *Let $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, and let $f_{\varepsilon} = f * P_{\varepsilon}$. Then the following hold:*

- (a) *If $1 \leq p < \infty$, then $f_{\varepsilon} \rightarrow f$ in $L^p(\mathbb{R}^n)$ as $\varepsilon \rightarrow 0^+$.*
- (b) *If $1 < p \leq \infty$, then $f_{\varepsilon} \rightarrow f$ almost everywhere as $\varepsilon \rightarrow 0^+$.*
- (c) *If $1 \leq p \leq \infty$, then $f * P_{\mathbf{t}}(x)$ is smooth in $(x, \mathbf{t}) \in \mathbb{R}^n \times (\mathbb{R}_+)^m$.*

Proof. We can write

$$\begin{aligned}
f * P_t(x) &= \int_{\mathbb{R}^m} f \left(x - \sum_{j=1}^m s_j e_j \right) P_{t_1}(s_1) \cdots P_{t_m}(s_m) ds \\
(3.11) \quad &= \int_{\mathbb{R}^m} f \left(x - \sum_{j=1}^n \left(s_j + \sum_{\mu=n+1}^m A_{\mu j} s_\mu \right) e_j \right) P_{t_1}(s_1) \cdots P_{t_m}(s_m) ds \\
&= \int_{\mathbb{R}^m} f \left(x - \sum_{j=1}^n s'_j e_j \right) \prod_{j=1}^n P_{t_j} \left(s'_j - \sum_{\mu=n+1}^m A_{\mu j} s'_\mu \right) \cdot \prod_{j=n+1}^m P_{t_j}(s'_j) ds' \\
&= \int_{\mathbb{R}^n} f(x - x') \pi(P_{t_1} \cdots P_{t_m})(x') dx' = f * \pi(P_{t_1} \cdots P_{t_m})(x),
\end{aligned}$$

using the coordinates transformations (3.8), where

$$(3.12) \quad \pi(P_{t_1} \cdots P_{t_m})(x') := \int_{\mathbb{R}^{m-n}} \prod_{j=1}^n P_{t_j} \left(s'_j - \sum_{\mu=n+1}^m A_{\mu j} s'_\mu \right) \cdot \prod_{j=n+1}^m P_{t_j}(s'_j) \frac{ds'_{n+1} \cdots ds'_m}{|\det(e_1, \dots, e_n)|}$$

for $x' = \sum_{j=1}^n s'_j e_j$. This function lies in $L^q(\mathbb{R}^n)$ for any $1 \leq q < \infty$, since

$$\begin{aligned}
&\|\pi(P_{t_1} \cdots P_{t_m})\|_{L^q(\mathbb{R}^n)} \\
(3.13) \quad &= \left(\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^{m-n}} \prod_{j=1}^n P_{t_j} \left(s'_j - \sum_{\mu=n+1}^m A_{\mu j} s'_\mu \right) \prod_{j=n+1}^m P_{t_j}(s'_j) ds'_{n+1} \cdots ds'_m \right|^q ds'_1 \cdots ds'_n \right)^{1/q} \\
&\leq \int_{\mathbb{R}^{m-n}} \left(\int_{\mathbb{R}^n} \left| \prod_{j=1}^n P_{t_j} \left(s'_j - \sum_{\mu=n+1}^m A_{\mu j} s'_\mu \right) \prod_{j=n+1}^m P_{t_j}(s'_j) \right|^q ds'_1 \cdots ds'_n \right)^{1/q} ds'_{n+1} \cdots ds'_m \\
&= (t_1 \cdots t_n)^{1/q-1} \|P_1\|_{L^q(\mathbb{R})}^n < \infty,
\end{aligned}$$

in view of the coordinates transformation (3.10) and Minkowski's inequality.

The case $q = \infty$ is obtained in the same way, replacing Minkowski's inequality by the elementary estimate

$$\|\pi(P_{t_1} \cdots P_{t_m})\|_{L^\infty(\mathbb{R}^n)} \lesssim \prod_{j=1}^n \|P_{t_j}\|_{L^\infty(\mathbb{R})} \lesssim (t_1 \cdots t_n)^{-1}.$$

Thus, the convolution in (3.11) is well defined.

The proof of convergence is similar to [42, Theorem 2 (c) in Chapter 3]. Take $\varphi(x) = \pi(P_{t_1} \cdots P_{t_m})(x)$ with $t_1 = \cdots = t_m = 1$. It is direct to check by definition (3.12) that

$$\varepsilon^{-n} \varphi(x/\varepsilon) = \pi(P_\varepsilon \cdots P_\varepsilon), \quad \int_{\mathbb{R}^n} \varphi(x) dx = 1.$$

Let $\omega_p(y) := \|f(\cdot - y) - f(\cdot)\|_{L^p(\mathbb{R}^n)}$ for $f \in L^p(\mathbb{R}^n)$. It is obviously bounded and is well known that $\omega_p(y)$ tends to 0 as $|y| \rightarrow 0$. Hence,

$$\begin{aligned}
\|f_\varepsilon - f\|_{L^p(\mathbb{R}^n)} &= \left(\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} (f(x - y) - f(x)) \varepsilon^{-n} \varphi(y/\varepsilon) dy \right|^p dx \right)^{1/p} \\
&\leq \int_{\mathbb{R}^n} \omega_p(y) \varepsilon^{-n} \varphi(y/\varepsilon) dy = \int_{\mathbb{R}^n} \omega_p(\varepsilon y) \varphi(y) dy \rightarrow 0,
\end{aligned}$$

as $\varepsilon \rightarrow 0$, by using Minkowski's inequality and Lebesgue's dominated convergence theorem.

(b) This can be proved as in [42, Chapter III, Theorem 2(b)], since

$$\sup_{\varepsilon > 0} |f * \pi(P_\varepsilon \cdots P_\varepsilon)(x)| \lesssim M_{\text{it}}(f)(x)$$

and M_{it} is bounded on $L^p(\mathbb{R}^n)$ for $1 < p \leq \infty$. The endpoint $p = 1$ is not included, since M_{it} is not of weak type $(1, 1)$. We omit the details.

(c) Recall that the Poisson kernel P_t satisfies the well-known estimates

$$(3.14) \quad |(t\partial_t)^a (t\partial_s)^b P_t(s)| \lesssim P_t(s),$$

for every $a, b \in \mathbb{N}_0$. By differentiating (3.12) and applying (3.14) to each Poisson factor, we obtain

$$t_1^{b_1} \cdots t_n^{b_n} t_1^{b'_1} \cdots t_m^{b'_m} \left| \partial_{s_1}^{b_1} \cdots \partial_{s_n}^{b_n} \partial_{t_1}^{b'_1} \cdots \partial_{t_m}^{b'_m} \pi(P_{t_1} \cdots P_{t_m}) \left(\sum_{j=1}^n s'_j e_j \right) \right| \lesssim \pi(P_{t_1} \cdots P_{t_m}) \left(\sum_{j=1}^n s'_j e_j \right),$$

for all $b_1, \dots, b_n, b'_1, \dots, b'_m \in \mathbb{N}_0$. Now fix a compact set $K \subset (\mathbb{R}_+)^m$. Then there exist $0 < \kappa < M < \infty$ such that $\kappa \leq t_j \leq M$, for all $\mathbf{t} = (t_1, \dots, t_m) \in K$. Since $P_t(s) \leq C_{\kappa, M} P_\kappa(s)$ for $\kappa \leq t \leq M$, we get

$$\left| \partial_{s_1}^{b_1} \cdots \partial_{s_n}^{b_n} \partial_{t_1}^{b'_1} \cdots \partial_{t_m}^{b'_m} \pi(P_{t_1} \cdots P_{t_m}) \left(\sum_{j=1}^n s'_j e_j \right) \right| \lesssim \pi(P_\kappa \cdots P_\kappa) \left(\sum_{j=1}^n s'_j e_j \right), \quad \mathbf{t} \in K,$$

where the implicit constant is independent of $x \in \mathbb{R}^n, \mathbf{t} \in K$. Consequently,

$$\left| \partial_{x_1}^{b_1} \cdots \partial_{x_n}^{b_n} \partial_{t_1}^{b'_1} \cdots \partial_{t_m}^{b'_m} \pi(P_{t_1} \cdots P_{t_m})(x) \right| \lesssim \pi(P_\kappa \cdots P_\kappa)(x), \quad x \in \mathbb{R}^n, \mathbf{t} \in K.$$

From this estimate together with (3.13), we can show that $f * P_{\mathbf{t}}(x)$ is smooth in $(x, \mathbf{t}) \in \mathbb{R}^n \times (\mathbb{R}_+)^m$ by repeated use of differentiation under the integral sign (cf. [12, Proposition 2.23]). $\square$

Proposition 3.4. *For $f \in L^p(\mathbb{R}^n)$ with $1 \leq p < \infty$, let $u(x, \mathbf{t}) := f * P_{\mathbf{t}}$. Then, for $b_1, \dots, b_m \in \mathbb{N}_0$,*

$$(3.15) \quad \begin{aligned} t_1^{b_1} \cdots t_m^{b_m} \left\| \nabla_1^{b_1} \cdots \nabla_m^{b_m} u(\cdot, \mathbf{t}) \right\|_{L^p(\mathbb{R}^n)} &\lesssim \|f\|_{L^p(\mathbb{R}^n)}, \\ t_1^{b_1} \cdots t_m^{b_m} \left| \nabla_1^{b_1} \cdots \nabla_m^{b_m} u(x, \mathbf{t}) \right| &\lesssim \frac{\|f\|_{L^p(\mathbb{R}^n)}}{|R(0, \mathbf{t})|^{1/p}}, \end{aligned}$$

for $(x, \mathbf{t}) \in \mathbb{R}^n \times (\mathbb{R}_+)^m$.

Proof. If we let $f_\varepsilon := f * P_\varepsilon$, then we can write $u(x, \mathbf{t}) = f_\varepsilon * P_{\mathbf{t}-\varepsilon}$ for small $\varepsilon > 0$ and f_ε is smooth by Proposition 3.3. So we may assume that f is smooth. The result will follow from (3.15) for f_ε and then taking limit $\varepsilon \rightarrow 0^+$. For fixed μ , we use coordinates $x = x^\perp + se_\mu$ as above. Then,

$$(3.16) \quad \begin{aligned} \|f *_\mu P_{t_\mu}\|_{L^p(\mathbb{R}^n)}^p &= \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \left| \int_{\mathbb{R}} f(x^\perp + (s-s')e_\mu) P_{t_\mu}(s') ds' \right|^p ds dx^\perp \\ &= \int_{\mathbb{R}^{n-1}} \|f(x^\perp + \cdot e_\mu) * P_{t_\mu}\|_{L^p(\mathbb{R})}^p dx^\perp \\ &\leq \int_{\mathbb{R}^{n-1}} \|f(x^\perp + \cdot e_\mu)\|_{L^p(\mathbb{R})}^p \|P_{t_\mu}\|_{L^1(\mathbb{R})}^p dx^\perp \\ &= \int_{\mathbb{R}^{n-1}} \|f(x^\perp + \cdot e_\mu)\|_{L^p(\mathbb{R})}^p dx^\perp = \|f\|_{L^p(\mathbb{R}^n)}^p \end{aligned}$$

by Young's inequality. Applying this inequality repeatedly, we see that $\|u(\cdot, \mathbf{t})\|_{L^p(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)}$. By replacing P_{t_μ} by $t_\mu^{b_\mu} \widetilde{\nabla}_\mu^{b_\mu} P_{t_\mu}$, the above argument gives us the first estimate in (3.15), since we also have $\|t_\mu^{b_\mu} \widetilde{\nabla}_\mu^{b_\mu} P_{t_\mu}\|_{L^1(\mathbb{R})} \lesssim 1$.

By Proposition 2.1 (2), $u(x + s'_\mu e_\mu, \mathbf{t}')$ is harmonic on $\{(s'_\mu, t'_\mu) : t'_\mu > 0\}$, which contains the disc $|s'_\mu + i(t_\mu - t'_\mu)| < t_\mu$. So we can apply the mean value formula to obtain

$$\begin{aligned} |u(x, \mathbf{t})|^p &= \left| \frac{1}{\pi t_\mu^2} \int_{|s'_\mu + i t'_\mu| < t_\mu} u(x + s'_\mu e_\mu, t_1, \dots, t_\mu + t'_\mu, \dots) ds'_\mu dt'_\mu \right|^p \\ &\leq \frac{1}{\pi t_\mu^2} \int_{|s'_\mu + i t'_\mu| < t_\mu} |u(x + s'_\mu e_\mu, t_1, \dots, t_\mu + t'_\mu, \dots)|^p ds'_\mu dt'_\mu \end{aligned}$$

by Hölder's inequality. Repeating this procedure, we deduce

(3.17)

$$\begin{aligned} |u(x, \mathbf{t})|^p &\leq \frac{1}{\pi^m t_1^2 \dots t_m^2} \int_{|s'_1 + i t'_1| < t_1} \dots \int_{|s'_m + i t'_m| < t_m} \left| u \left(x + \sum_{\mu=1}^m s'_\mu e_\mu, \mathbf{t} + \mathbf{t}' \right) \right|^p ds'_m dt'_m \dots ds'_1 dt'_1 \\ &\leq \frac{1}{\pi^m t_1^2 \dots t_m^2} \int_{-t_1}^{t_1} \dots \int_{-t_m}^{t_m} \int_{-t_1}^{t_1} \dots \int_{-t_m}^{t_m} \left| u \left(x + \sum_{\mu=1}^m s'_\mu e_\mu, \mathbf{t} + \mathbf{t}' \right) \right|^p dt'_m \dots dt'_1 ds'_m \dots ds'_1. \end{aligned}$$

Note that

$$\begin{aligned} \frac{1}{t_1 \dots t_m} \int_{-t_1}^{t_1} \dots \int_{-t_m}^{t_m} \left| u \left(x + \sum_{\mu=1}^m s'_\mu e_\mu, \mathbf{t} + \mathbf{t}' \right) \right|^p ds'_m \dots ds'_1 &= |u(\cdot, \mathbf{t} + \mathbf{t}')|^p * \chi_{\mathbf{t}}(x) \\ &\lesssim \frac{1}{|R(x, \gamma_0^{-1} \mathbf{t})|} \int_{R(x, \gamma_0^{-1} \mathbf{t})} |u(x', \mathbf{t} + \mathbf{t}')|^p dx' \\ &\lesssim \frac{1}{|R(x, \gamma_0^{-1} \mathbf{t})|} \int_{\mathbb{R}^n} |u(x', \mathbf{t} + \mathbf{t}')|^p dx' \\ &\lesssim \frac{1}{|R(x, \mathbf{t})|} \|f\|_{L^p(\mathbb{R}^n)}^p, \end{aligned}$$

by applying Proposition 3.2 and the first estimate in (3.15). Substituting this estimate into (3.17), we obtain the second inequality in (3.15) for $b_1 = \dots = b_m = 0$.

For the general case, note that each component of $\nabla_1^{b_1} \dots \nabla_m^{b_m} u(x + s'_\mu e_\mu, \mathbf{t}')$ is still harmonic on $\{(s'_\mu, t'_\mu) : t'_\mu > 0\}$. The result follows from the same argument by using the first estimate in (3.15). $\square$

Corollary 3.1. Fix $\varepsilon > 0$ and let $f_\varepsilon := f * P_\varepsilon$ for real-valued $f \in L^1(\mathbb{R}^n)$. Then, $U(x, t_\mu) = (f_\varepsilon *_\mu P_{t_\mu}(x))^2$ satisfies the conditions in Proposition 2.3.

Proof. For condition (1), write $U(x, t_\mu) = u^2(x, t_\mu)$ with $u(x, t_\mu) := f_\varepsilon *_\mu P_{t_\mu}(x)$. It is smooth on $\mathbb{R}_+^{n+1}$ since f_ε is smooth and every derivative of f_ε is integrable by Proposition 3.4. Since $u(\cdot, t_\mu), \nabla_\mu u(\cdot, t_\mu) \in L^1 \cap L^\infty(\mathbb{R}^n)$ with norms only depending on ε and $\|f\|_1$ by Proposition 3.4, we see that $U(\cdot, t_\mu), \nabla_\mu U(\cdot, t_\mu) = 2u(\cdot, t_\mu) \nabla_\mu u(\cdot, t_\mu) \in L^1 \cap L^\infty(\mathbb{R}^n)$. It is similar to see $\nabla_\mu^2 U(\cdot, t_\mu) \in L^1 \cap L^\infty(\mathbb{R}^n)$. Consequently, $\nabla_\mu^a U \in L^1(\mathbb{R}^n \times [0, T])$ for any $T > 0$. On the other hand, it is standard to see that for $x = x^\perp + s e_\mu$,

$$X_\mu U(x, t_\mu) = 2u(x, t_\mu) \cdot \int_{\mathbb{R}} f_\varepsilon(x^\perp + (s - s') e_\mu) \partial_{s'} P_{t_\mu}(s') ds' \rightarrow 0,$$

for almost all $x^\perp \in e_\mu^\perp$ as $s \rightarrow \pm\infty$, since $f_\varepsilon(x^\perp + \cdot e_\mu) \in L^1(\mathbb{R})$ for almost all $x^\perp \in e_\mu^\perp$ and u is bounded.

For condition (2), apply the estimates (3.15) to $u(x, t_\mu) = f * P_\varepsilon *_\mu P_{t_\mu}(x)$ to get

$$\|t_\mu \partial_{t_\mu} U(\cdot, t_\mu)\|_{L^1(\mathbb{R}^n)} \lesssim \|u(\cdot, t_\mu)\|_{L^\infty(\mathbb{R}^n)} \|t_\mu \partial_{t_\mu} u(\cdot, t_\mu)\|_{L^1(\mathbb{R}^n)} \lesssim \frac{\|f\|_{L^1(\mathbb{R}^n)}^2}{\varepsilon^{n-1}(\varepsilon + t_\mu)} \rightarrow 0$$

as $t_\mu \rightarrow +\infty$. Similarly, $\|U(\cdot, t_\mu)\|_{L^1(\mathbb{R}^n)} \rightarrow 0$ as $t_\mu \rightarrow +\infty$.

For condition (3), note that $u(x, 0) = f_\varepsilon$. Write $\omega_1(x) := \|f_\varepsilon(\cdot - x) - f_\varepsilon(\cdot)\|_{L^1(\mathbb{R}^n)}$. It is well known that $\omega_1(x) \rightarrow 0$ as $|x| \rightarrow 0$. Hence,

$$\begin{aligned} \|u(\cdot, t_\mu) - u(\cdot, 0)\|_{L^1(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}} [f_\varepsilon(x^\perp + (s - s')e_\mu) - f_\varepsilon(x^\perp + se_\mu)] P_{t_\mu}(s') ds' \right| dx^\perp ds \\ &\leq \int_{\mathbb{R}} \omega_1(s'e_\mu) P_{t_\mu}(s') ds' = \int_{\mathbb{R}} \omega_1(t_\mu s'e_\mu) P_1(s') ds' \rightarrow 0, \end{aligned}$$

as $t_\mu \rightarrow 0$, by rescaling and using Lebesgue's dominated convergence theorem (cf. [42, P. 63]). Note that $\|t_\mu \partial_{t_\mu} u(\cdot, t_\mu)\|_{L^1(\mathbb{R}^n)} \rightarrow 0$ as $t_\mu \rightarrow 0$ follows from the same argument, since

$$t_\mu \partial_{t_\mu} u(x, t_\mu) = \int_{\mathbb{R}} [f_\varepsilon(x - s'e_\mu) - f_\varepsilon(x)] t_\mu \partial_{t_\mu} P_{t_\mu}(s') ds'$$

by $\int_{\mathbb{R}} t_\mu \partial_{t_\mu} P_{t_\mu}(s') ds' = 0$. The result for $U(x, t_\mu) = u^2(x, t_\mu)$ follows from the fact that $|u(x, t_\mu)|$ has an upper bound only depending on ε . $\square$

Proposition 3.5. *Let $F \in H^p(T_\Omega)$ with $p \geq 1$. Then for $(x, \mathbf{r}) \in \mathbb{R}^n \times (\mathbb{R}_+)^m$, we have*

$$(3.18) \quad |F(x + \mathbf{i}\pi(\mathbf{r}))| \lesssim \frac{\|F\|_{H^p(T_\Omega)}}{|R(0, \mathbf{r})|^{1/p}}.$$

Proof. Given $y \in \Omega$, write $y = \pi(\mathbf{r}) = \sum_{\mu=1}^m r_\mu e_\mu \in \Omega$ for some $\mathbf{r} \in (\mathbb{R}_+)^m$. Note that the disc $\{x + \mathbf{i}y + (s_\mu + \mathbf{i}t_\mu)e_\mu : |s_\mu + \mathbf{i}t_\mu| < r_\mu\}$ is contained in the tube domain T_Ω . This is because $r_\mu + t_\mu > r_\mu - r_\mu = 0$. Thus, we can apply the mean value formula for a smaller disc in the complex line $\mathbb{C}e_\mu$ to get

$$F(x + \mathbf{i}y) = \frac{1}{\pi \gamma_0^2 r_\mu^2} \int_{|s_\mu + \mathbf{i}t_\mu| < \gamma_0 r_\mu} F(x + \mathbf{i}y + (s_\mu + \mathbf{i}t_\mu)e_\mu) ds_\mu dt_\mu,$$

where γ_0 is defined as in Proposition 3.1. Repeating this procedure, we get

$$\begin{aligned} |F(x + \mathbf{i}y)| &\leq \int_{|s_1 + \mathbf{i}t_1| < \gamma_0 r_1} \cdots \int_{|s_m + \mathbf{i}t_m| < \gamma_0 r_m} \frac{|F(x + \mathbf{i}y + \sum_{\mu=1}^m (s_\mu + \mathbf{i}t_\mu)e_\mu)|}{\pi^m \gamma_0^{2m} r_1^2 \cdots r_m^2} ds_m dt_m \cdots ds_1 dt_1 \\ &\leq \int_{-\gamma_0 r_1}^{\gamma_0 r_1} \cdots \int_{-\gamma_0 r_m}^{\gamma_0 r_m} \int_{-\gamma_0 r_1}^{\gamma_0 r_1} \cdots \int_{-\gamma_0 r_m}^{\gamma_0 r_m} \frac{|F(x + \mathbf{i}y + \sum_{\mu=1}^m (s_\mu + \mathbf{i}t_\mu)e_\mu)|}{\pi^m \gamma_0^{2m} r_1^2 \cdots r_m^2} dt_m \cdots dt_1 ds_m \cdots ds_1. \end{aligned}$$

Note that

$$\begin{aligned} &\int_{-\gamma_0 r_1}^{\gamma_0 r_1} \cdots \int_{-\gamma_0 r_m}^{\gamma_0 r_m} \frac{|F(x + \mathbf{i}y + \sum_{\mu=1}^m s_\mu e_\mu + \mathbf{i} \sum_{\mu=1}^m t_\mu e_\mu)|}{\gamma_0^m r_1 \cdots r_m} dt_m \cdots dt_1 \\ &= \left| F\left(x + \sum_{\mu=1}^m s_\mu e_\mu + \mathbf{i} \cdot\right) \right| * \chi_{\gamma_0 \mathbf{r}}(y) \lesssim \frac{1}{|R(y, \mathbf{r})|} \int_{R(y, \mathbf{r})} \left| F\left(x + \sum_{\mu=1}^m s_\mu e_\mu + \mathbf{i}y'\right) \right| dy', \end{aligned}$$

by applying Proposition 3.2, where $R(y, \mathbf{r}) \subset \Omega$. Apply the same procedure for the integral over $s_1, \dots, s_m$ to get

$$\begin{aligned}
 |F(x + \mathbf{i}y)| &\lesssim \frac{1}{|R(0, \mathbf{r})|^2} \int_{R(x, \mathbf{r})} \int_{R(y, \mathbf{r})} |F(x' + \mathbf{i}y')| dy' dx' \\
 (3.19) \quad &\leq \frac{1}{|R(0, \mathbf{r})|^{1+1/p}} \int_{R(y, \mathbf{r})} \left(\int_{\mathbb{R}^n} |F(x' + \mathbf{i}y')|^p dx' \right)^{1/p} dy' \\
 &\lesssim \frac{1}{|R(0, \mathbf{r})|^{1/p}} \|F\|_{H^p(T_\Omega)}.
 \end{aligned}$$

This concludes the proof of (3.18). $\square$

3.3. Proof of iterated Poisson integral formula and harmonic control. Stein and Weiss [46, Section 5 of Chapter 3] established the following boundary convergence theorem for $H^p(T_\Omega)$ functions.

Theorem 3.1. *Suppose $F \in H^p(T_\Omega)$, $1 \leq p < \infty$, where Ω is a regular open cone in $\mathbb{R}^n$. Then, if Ω_1 is any cone whose closure is contained in $\Omega \cup \{0\}$, we have*

- (a) $\lim_{y \in \Omega_1, y \rightarrow 0} F(x + \mathbf{i}y) = F^b(x)$ exists for almost every $x \in \mathbb{R}^n$;
- (b) $\int_{\mathbb{R}^n} |F(x + \mathbf{i}y) - F^b(x)|^p dx \rightarrow 0$ as $y \in \Omega_1$ tends to 0.

Moreover, if Ω is a polyhedral cone, Ω_1 can be replaced by Ω .

Boundary limits of functions in $H^p(T_\Omega)$ have the following Fourier spectrum characterization (cf. [31, Theorem 2.1] and [46, Theorem 3.1 in Chapter 3] for $p = 2$).

Theorem 3.2. *Let Ω be a regular open cone in $\mathbb{R}^n$, and let $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq 2$. Then f is the boundary limit function of some function $F(x + \mathbf{i}y) \in H^p(T_\Omega)$ if and only if $\text{supp } \widehat{f} \subset \Omega^*$, where $\widehat{f}$ is the Fourier transform of f . Moreover, if the condition is satisfied, then*

$$(3.20) \quad F(x + \mathbf{i}y) = \int_{\Omega^*} e^{2\pi i(x + \mathbf{i}y) \cdot \xi} \widehat{f}(\xi) d\xi.$$

Corollary 3.2. *For two polyhedral cones Ω and Ω' in $\mathbb{R}^n$ such that $\overline{\Omega} \subset \Omega' \cup \{0\}$, one has $H^1(T_{\Omega'}) \subsetneq H^1(T_\Omega)$.*

Proof. Since $T_\Omega \subset T_{\Omega'}$, $H^1(T_{\Omega'}) \subset H^1(T_\Omega)$ in view of definition (1.1). Note that $(\Omega')^* \subsetneq \Omega^*$ by definition. Let ψ be a smooth function supported in a bounded subset of Ω^* , but not contained in $(\Omega')^*$. Set $F_0(x + \mathbf{i}y) = \int_{\Omega^*} e^{2\pi i(x + \mathbf{i}y) \cdot \xi} \psi(\xi) d\xi$. It is direct to see that $F_0 \in H^1(T_\Omega)$, but $F_0 \notin H^1(T_{\Omega'})$ by Theorem 3.2. $\square$

Proof of Theorem 1.1. For each fixed $y \in \Omega$, there exists $\delta = \delta_y > 0$ such that $y \cdot \xi \geq \delta|\xi|$ for any $\xi \in \Omega^*$ (cf. [45]). Therefore, for $z = x + \mathbf{i}y$,

$$\left| \chi_{\Omega^*}(\xi) e^{2\pi i z \cdot \xi} \widehat{F^b}(\xi) \right| = \chi_{\Omega^*}(\xi) |\widehat{F^b}(\xi)| e^{-2\pi y \cdot \xi} \leq \chi_{\Omega^*}(\xi) |\widehat{F^b}(\xi)| e^{-2\pi \delta |\xi|}.$$

Since $F^b \in L^1(\mathbb{R}^n)$ by Theorem 3.1, we have $\widehat{F^b} \in L^\infty(\mathbb{R}^n)$. Hence, $\chi_{\Omega^*} e^{-2\pi y \cdot \xi} \widehat{F^b} \in L^1(\mathbb{R}^n)$. By the Fourier spectrum characterization in Theorem 3.2, we obtain

$$(3.21) \quad \widehat{F(\cdot + \mathbf{i}y)}(\xi) = \chi_{\Omega^*}(\xi) e^{-2\pi y \cdot \xi} \widehat{F^b}(\xi), \quad \text{a.e. } \xi \in \mathbb{R}^n.$$

On the other hand, the one-dimensional Poisson kernel $P_t(s) = \frac{1}{\pi} \frac{t}{t^2 + s^2}$ satisfies $\widehat{P}_t(\tau) = e^{-2\pi t|\tau|}$. Thus, for $f \in L^1(\mathbb{R}^n)$,

$$\begin{aligned} \widehat{f *_{\mu} P_{t_{\mu}}}(\xi) &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}} f(x - se_{\mu}) P_{t_{\mu}}(s) ds \right) e^{-2\pi i x \cdot \xi} dx \\ &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}} f(x') e^{-2\pi i x' \cdot \xi} dx' \right) e^{-2\pi i s e_{\mu} \cdot \xi} P_{t_{\mu}}(s) ds \\ &= \widehat{f}(\xi) \widehat{P_{t_{\mu}}}(e_{\mu} \cdot \xi) = \widehat{f}(\xi) e^{-2\pi t_{\mu} |e_{\mu} \cdot \xi|} \end{aligned}$$

by using Fubini's theorem and changing variables. Iterating this identity gives

$$(3.22) \quad \widehat{f * P_{\mathbf{t}}}(\xi) = \widehat{f}(\xi) e^{-2\pi \sum_{\mu=1}^m t_{\mu} |e_{\mu} \cdot \xi|}.$$

Since $\text{supp } \widehat{F^b} \subset \Omega^*$ by Theorem 3.2, and $e_{\mu} \cdot \xi \geq 0$ for any $\xi \in \Omega^*$, we have

$$\sum_{\mu=1}^m t_{\mu} |e_{\mu} \cdot \xi| = \sum_{\mu=1}^m t_{\mu} e_{\mu} \cdot \xi = \pi(\mathbf{t}) \cdot \xi$$

on $\text{supp } \widehat{F^b}$. Apply (3.22) to F^b to get

$$\widehat{F^b * P_{\mathbf{t}}}(\xi) = \chi_{\Omega^*}(\xi) \widehat{F^b}(\xi) e^{-2\pi \pi(\mathbf{t}) \cdot \xi}.$$

Comparing this identity with (3.21) for $\widehat{F(\cdot + i\mathbf{y})}$, with $y = \pi(\mathbf{t})$, we obtain

$$F(x + i\pi(\mathbf{t})) = F^b * P_{\mathbf{t}}(x)$$

as L^1 -functions. Since $F(x + i\pi(\mathbf{t}))$ is smooth in x as a holomorphic function and $F^b * P_{\mathbf{t}}$ is also smooth by Proposition 3.3, the equality holds for every $x \in \mathbb{R}^n$. This proves the theorem. $\square$

Proposition 3.6. *Suppose $F \in H^1(T_{\Omega})$. Then for $0 < q < 1$, we have*

$$(3.23) \quad |F(x + i\pi(\mathbf{t} + \boldsymbol{\varepsilon}))|^q \leq |f_{\boldsymbol{\varepsilon}}|^q * P_{\mathbf{t}}(x),$$

for every $(x, \mathbf{t}) \in \mathbb{R}^n \times (\mathbb{R}_+)^m$ and $\boldsymbol{\varepsilon} > 0$, where $f_{\boldsymbol{\varepsilon}}(x) := F(x + i\pi(\boldsymbol{\varepsilon}))$.

Proof. We first prove a one-parameter estimate. For fixed $\mu \in \{1, \dots, m\}$, let $\mathbf{b}_{\mu}$ denote the μ -th standard basis vector of $\mathbb{R}^m$ and let $\mathbf{t}^{(\mu)} := \mathbf{t} - t_{\mu} \mathbf{b}_{\mu}$ for $\mathbf{t} = (t_1, \dots, t_m) \in \mathbb{R}_+^m$. Set

$$(3.24) \quad g_{\mathbf{t}, \mu}(x) := F(x + i\pi(\mathbf{t}^{(\mu)} + \boldsymbol{\varepsilon})).$$

Since $\mathbf{t}^{(\mu)} + \boldsymbol{\varepsilon} \in (\mathbb{R}_+)^m$, we have $g_{\mathbf{t}, \mu} \in L^1(\mathbb{R}^n) \cap L^{\infty}(\mathbb{R}^n)$ by the assumption $F \in H^1(T_{\Omega})$ and Proposition 3.5. By Theorem 1.1, $g_{\mathbf{t}, \mu} = F^b * P_{\mathbf{t}^{(\mu)} + \boldsymbol{\varepsilon}}$. Using the semigroup property of the one-dimensional Poisson kernels and the commutativity of partial convolutions, we obtain

$$(3.25) \quad g_{\mathbf{t}, \mu} *_{\mu} P_r = F^b * P_{\mathbf{t}^{(\mu)} + \boldsymbol{\varepsilon}} *_{\mu} P_r = F^b * P_{\mathbf{t}^{(\mu)} + r\mathbf{b}_{\mu} + \boldsymbol{\varepsilon}} = F(\cdot + i\pi(\mathbf{t}^{(\mu)} + r\mathbf{b}_{\mu} + \boldsymbol{\varepsilon})).$$

The equalities above first hold as L^1 -functions and hence hold everywhere by continuity.

Now write $x = x^{\perp} + \sigma e_{\mu}$, with $x^{\perp} \in e_{\mu}^{\perp}$, $\sigma \in \mathbb{R}$. Since $g_{\mathbf{t}, \mu} \in L^1(\mathbb{R}^n)$, Fubini's theorem and $g_{\mathbf{t}, \mu} \in L^{\infty}(\mathbb{R}^n)$ imply that for almost every $x^{\perp} \in e_{\mu}^{\perp}$,

$$(3.26) \quad g_{\mathbf{t}, \mu, x^{\perp}}(\sigma) := g_{\mathbf{t}, \mu}(x^{\perp} + \sigma e_{\mu})$$

belongs to $L^1(\mathbb{R}) \cap L^{\infty}(\mathbb{R})$. For such an $x^{\perp}$, define

$$(3.27) \quad h_{x^{\perp}}(z) := F(x^{\perp} + ze_{\mu} + i\pi(\mathbf{t}^{(\mu)} + \boldsymbol{\varepsilon})), \quad z \in \mathbb{C}_+.$$

Since $\pi(\mathbf{t}^{(\mu)} + \varepsilon) + (\operatorname{Im} z)e_\mu \in \Omega$, Proposition 2.1(1) shows that $h_{x^\perp}$ is holomorphic on $\mathbb{C}_+$. Moreover,

$$(3.28) \quad h_{x^\perp}(\sigma + ir) = (g_{\mathbf{t}, \mu, x^\perp} * P_r)(\sigma),$$

where $\sigma \in \mathbb{R}, r > 0$, by (3.25). Consequently,

$$(3.29) \quad \sup_{r>0} \int_{\mathbb{R}} |h_{x^\perp}(\sigma + ir)| d\sigma \leq \sup_{r>0} \|g_{\mathbf{t}, \mu, x^\perp} * P_r\|_{L^1(\mathbb{R})} \leq \|g_{\mathbf{t}, \mu, x^\perp}\|_{L^1(\mathbb{R})} < \infty.$$

Thus $h_{x^\perp} \in H^1(\mathbb{C}_+)$ and $h_{x^\perp}^b = g_{\mathbf{t}, \mu, x^\perp}$ a.e. on $\mathbb{R}$.

Now applying the classical one-dimensional subharmonic inequality for $H^1(\mathbb{C}_+)$ -functions [42, Corollary 2, p. 221] to $h_{x^\perp}$, we get that for $0 < q < 1$,

$$(3.30) \quad |h_{x^\perp}(\sigma + ir)|^q \leq (|g_{\mathbf{t}, \mu, x^\perp}|^q * P_r)(\sigma),$$

where $\sigma \in \mathbb{R}, r > 0$. Taking $r = t_\mu$, we get

$$(3.31) \quad |F(x + \mathbf{i}\pi(\mathbf{t} + \varepsilon))|^q \leq \left| F(\cdot + \mathbf{i}\pi(\mathbf{t}^{(\mu)} + \varepsilon)) \right|^q *_{\mu} P_{t_\mu}(x)$$

for almost every $x \in \mathbb{R}^n$. By continuity, the inequality holds for every $x \in \mathbb{R}^n$.

We now iterate (3.31). Using the positivity of the Poisson kernels, we obtain

$$(3.32) \quad \begin{aligned} |F(x + \mathbf{i}\pi(\mathbf{t} + \varepsilon))|^q &\leq \left| F(\cdot + \mathbf{i}\pi(\mathbf{t}^{(1)} + \varepsilon)) \right|^q *_1 P_{t_1}(x) \\ &\leq \left| F(\cdot + \mathbf{i}\pi(\mathbf{t}^{(1)} - t_2 \mathbf{b}_2 + \varepsilon)) \right|^q *_2 P_{t_2} *_1 P_{t_1}(x) \\ &\leq \cdots \leq |F(\cdot + \mathbf{i}\pi(\varepsilon))|^q *_{t_m} P_{t_m} \cdots *_1 P_{t_1}(x). \end{aligned}$$

Since the partial convolutions commute and $f_\varepsilon(x) = F(x + \mathbf{i}\pi(\varepsilon))$, we conclude the resulting inequality. $\square$

Remark 3.2. The iterated Poisson integral formula in Theorem 1.1 and subharmonic inequality 3.6 can also be proved by using the maximum principle as in [47].

Proposition 3.7. Assume that one vector in the generating system $\mathcal{E} = \{e_1, \dots, e_m\}$ of a polyhedral cone Ω , say e_1 , is not parallel to any vector in the generating system $\mathcal{E}' = \{e'_1, \dots, e'_{m'}\}$ of another polyhedral cone Ω' . Then, $H_{\max; \mathcal{E}}^1(\mathbb{R}^n) \neq H_{\max; \mathcal{E}'}^1(\mathbb{R}^n)$.

Proof. Since e_1 is not parallel to any e'_j , the hyperplane $e_1^\perp$ is not contained in any $(e'_j)^\perp$. Hence we can choose a point $\xi_0 \in e_1^\perp$ such that $e'_j \cdot \xi_0 \neq 0$, $j = 1, \dots, m'$. Choose a nonzero function $\psi \in C_c^\infty(\mathbb{R}^n)$ supported in a sufficiently small ball around ξ_0 such that $|e'_j \cdot \xi| \geq c > 0$, for $\xi \in \operatorname{supp} \psi, j = 1, \dots, m'$, and such that the restriction of ψ to $e_1^\perp$ is not identically zero. Set $f := \check{\psi}$. Then, $f \in \mathcal{S}(\mathbb{R}^n)$.

Let $u(x, \mathbf{t}) = f *'_1 P_{t_1} \cdots *'_{m'} P_{t_{m'}}(x)$ for $\mathbf{t} \in (\mathbb{R}_+)^{m'}$, where $*'_\mu$ is the partial convolution on $\mathbb{R}^n$ along the line e'_μ . By (3.22), we have

$$\widehat{u(\cdot, \mathbf{t})}(\xi) = e^{-2\pi \sum_{j=1}^{m'} t_j |e'_j \cdot \xi|} \psi(\xi).$$

Since $|e'_j \cdot \xi| \geq c$ on $\operatorname{supp} \psi$, by integration by parts in the Fourier integral, there exists $c_0 > 0$ such that

$$|u(x, \mathbf{t})| \lesssim e^{-c_0 \sum_{j=1}^{m'} t_j} (1 + |x|)^{-(n+1)}.$$

If $(x', \mathbf{t}) \in \Gamma_{\beta, \mathcal{E}'}(x)$, the non-tangential region associated to $\mathcal{E}'$, then $|x' - x| \leq C_\beta \sum_{j=1}^{m'} t_j$ for some constant $C_\beta > 0$. If $\sum_{j=1}^{m'} t_j \leq |x|/(2C_\beta)$, then $|x'| \geq |x|/2$, and hence $|u(x', \mathbf{t})| \lesssim (1 + |x|)^{-(n+1)}$. If $\sum_{j=1}^{m'} t_j > |x|/(2C_\beta)$, then $|u(x', \mathbf{t})| \lesssim e^{-c_1 |x|}$ for some constant $c_1 > 0$. Therefore,

$$N_{\mathcal{E}'}^\beta(f)(x) \lesssim (1 + |x|)^{-(n+1)} + e^{-c_1 |x|}.$$

We obtain $N_{\mathcal{E}'}^\beta(f) \in L^1(\mathbb{R}^n)$, and so $f \in H_{\max; \mathcal{E}'}^1(\mathbb{R}^n)$.

To show $f \notin H_{\max; \mathcal{E}}^1(\mathbb{R}^n)$, set $h_a(s) := f(a + se_1)$ for $a \in e_1^\perp, s \in \mathbb{R}$. Then, $h_a \in \mathcal{S}(\mathbb{R})$. Since e_1 is one of the generating directions of Ω , $N_{\mathcal{E}}^\beta(f)$ controls the 1-dimensional maximal function along each line $a + \mathbb{R}e_1$. More explicitly,

$$(3.33) \quad N_{\mathcal{E}}^\beta(f)(a + se_1) \geq \sup_{r>0} |h_a * P_r(s)|.$$

This follows by taking the e_1 -parameter equal to r and letting the remaining parameters tend to 0^+ in the iterated Poisson integral. Now for $a \in e_1^\perp$, define $G(a) := \int_{\mathbb{R}} f(a + se_1) ds$. The Fourier transform of G on the hyperplane $e_1^\perp$ is $\widehat{G}(\eta) = \widehat{f}(\eta) = \psi(\eta)$, for $\eta \in e_1^\perp$. Since $\psi|_{e_1^\perp} \neq 0$, we have $G \neq 0$. Hence the set $E := \{a \in e_1^\perp : G(a) \neq 0\}$ has positive measure. Consequently, for $a \in E$, $\int_{\mathbb{R}} h_a(s) ds = G(a) \neq 0$. So we must have $\int_{\mathbb{R}} \sup_{r>0} |h_a * P_r(s)| ds = +\infty$ for $a \in E$. Otherwise, the classical maximal-function characterization would give $h_a \in H^1(\mathbb{R})$, and so $\int_{\mathbb{R}} h_a(s) ds = 0$ for $a \in E$, which is a contradiction. Therefore, by (3.33) and Tonelli's theorem,

$$\int_{\mathbb{R}^n} N_{\mathcal{E}}^\beta(f)(x) dx \geq \int_E \int_{\mathbb{R}} \sup_{r>0} |h_a * P_r(s)| ds da = +\infty.$$

Thus $N_{\mathcal{E}}^\beta(f) \notin L^1(\mathbb{R}^n)$, and hence $f \notin H_{\max; \mathcal{E}}^1(\mathbb{R}^n)$. The proof is complete. $\square$

4. THE MAXIMAL FUNCTION CHARACTERIZATION

4.1. Reduction of the Fefferman-Stein type good- λ inequality to an integral estimate.

Theorem 4.1. *Let $f \in L^1(\mathbb{R}^n)$. If β is chosen sufficiently large and $N^\beta(f) \in L^1(\mathbb{R}^n)$, then the Fefferman-Stein type good- λ inequality (1.14) holds.*

By Lemma 3.1, the norms $\|N^\beta(f)\|_{L^1(\mathbb{R}^n)}$ are equivalent for different $\beta > 0$. The following is a direct corollary of this theorem.

Corollary 4.1. *For every fixed $\beta > 0$, $\|S(f)\|_{L^1(\mathbb{R}^n)} \lesssim \|N^\beta(f)\|_{L^1(\mathbb{R}^n)}$.*

To prove the Fefferman-Stein type good- λ inequality in Theorem 4.1, let

$$E_\beta(\lambda) := \{x \in \mathbb{R}^n : N^\beta(f)(x) \leq \lambda\},$$

for $f \in L^1(\mathbb{R}^n)$ with $N^\beta(f) \in L^1(\mathbb{R}^n)$ and $\lambda \geq 0$, and let

$$A_\beta(\lambda) := \left\{x \in \mathbb{R}^n : M_{\text{it}}(\chi_{E_\beta(\lambda)^c})(x) \leq \frac{1}{10C_0}\right\},$$

where C_0 is given by (3.1). By definition, $E_\beta(\lambda)^c \subset A_\beta(\lambda)^c$, and so $A_\beta(\lambda) \subset E_\beta(\lambda)$, up to a set of measure zero. Moreover,

$$|A_\beta(\lambda)^c| \lesssim \|M_{\text{it}}(\chi_{E_\beta(\lambda)^c})\|_{L^2}^2 \leq C|E_\beta(\lambda)^c|,$$

by the L^2 -boundedness of iterated maximal function, where C is independent of β . Therefore, we have

$$(4.1) \quad \begin{aligned} |\{x \in \mathbb{R}^n : S(f)(x) > \lambda\}| &\leq |\{x \in A_\beta(\lambda)^c : S(f)(x) > \lambda\}| + |\{x \in A_\beta(\lambda) : S(f)(x) > \lambda\}| \\ &\leq C|E_\beta(\lambda)^c| + \frac{1}{\lambda^2} \int_{A_\beta(\lambda)} S(f)^2(x) dx. \end{aligned}$$

Consider the domains (see Figure 2)

$$(4.2) \quad W_\beta := \bigcup_{x \in A_\beta(\lambda)} \Gamma(x), \quad \text{and} \quad \widetilde{W}_\beta := \bigcup_{x \in E_\beta(\lambda)} \Gamma_\beta(x).$$



(2) If we choose β sufficiently large, then there is a constant $C_1 \in (0, 9/10)$ such that for any $(x, \mathbf{t}) \in \widetilde{W}_\beta^c = (\mathbb{R}^n \times (\mathbb{R}_+)^m) \setminus \widetilde{W}_\beta$, we have $P_{\mathbf{t}}(\chi_{E_\beta(\lambda)})(x) \leq C_1$.

$$P_{\mathbf{t}}(\chi_{E_{\beta}(\lambda)^c})(x) \leq C_0 M_{\text{it}}(\chi_{E_{\beta}(\lambda)^c})(x') \leq C_0 \frac{1}{10C_0} = \frac{1}{10},$$

(2) Let $(x, \mathbf{t}) \in \widetilde{W}_\beta^c$. Given $\mathbf{s} \in \mathbb{R}^m$, since $R(0, \beta \mathbf{t}) = \pi(R(0, \beta \mathbf{t}))$, it is clear that if $\pi(\mathbf{s}) \notin R(0, \beta \mathbf{t})$, then $\mathbf{s} \notin \widetilde{R}(0, \beta \mathbf{t})$. Consequently, $|s_\mu| \geq \beta t_\mu$ holds at least for one μ . Thus,

for some constant $C > 0$ independent of $\mathbf{t}$, as $\beta \rightarrow +\infty$. Thus, by choosing sufficiently large β , we can find a constant $C_1 \in (0, 9/10)$ such that $P_{\mathbf{t}}(\chi_{E_{\beta}(\lambda)})(x) \leq C_1$ for any $(x, \mathbf{t}) \in \widetilde{W}_{\beta}^c$. The proof is complete. $\square$

$$(4.5) \quad U_1(x, \mathbf{t}) := f *_1 P_{t_1} *_2 \tilde{\nabla}_2 P_{t_2} * \cdots *_m \tilde{\nabla}_m P_{t_m}(x).$$

Let ϕ be an increasing non-negative $\mathcal{C}^2[0, 1]$ function such that

$$(4.6) \quad \phi(r) = 1 \quad \text{if } r \geq \frac{9}{10}, \quad \phi(r) = 0 \quad \text{in a neighborhood of } [0, C_1]$$

(cf. [33, 39]). Observe that by Proposition 4.1 (1), for $(x, \mathbf{t}) \in W_\beta$, we have $P_{\mathbf{t}}(\chi_{E_\beta(\lambda)})(x) \geq \frac{9}{10}$. Therefore, $\phi(P_{\mathbf{t}}(\chi_{E_\beta(\lambda)})(x)) = 1$ and so

$$(4.7) \quad \begin{aligned} \int_{A_\beta(\lambda)} S(f)^2(x) dx &= \int_{A_\beta(\lambda)} \int_{\Gamma(x)} \left| f * \tilde{\nabla}_1 P_{t_1} * \cdots * \tilde{\nabla}_m P_{t_m} \right|^2(x') \frac{\mathbf{t} d\mathbf{t} dx'}{|R(0, \mathbf{t})|} dx \\ &\lesssim \int_{W_\beta} \left| f * \tilde{\nabla}_1 P_{t_1} * \cdots * \tilde{\nabla}_m P_{t_m} \right|^2(x') \mathbf{t} d\mathbf{t} dx' \\ &= \int_{W_\beta} |\nabla_1 U_1(x, \mathbf{t})|^2 \mathbf{t} d\mathbf{t} dx \\ &\leq \int_{\mathbb{R}^n \times (\mathbb{R}_+)^m} |\nabla_1 U_1(x, \mathbf{t})|^2 \phi^2(P_{\mathbf{t}}(\chi_{E_\beta(\lambda)})(x)) \mathbf{t} d\mathbf{t} dx. \end{aligned}$$

4.2. Estimates of integrals by differential identities. To prove the Fefferman-Stein type inequality of Theorem 4.1, we need to estimate the integral in the R.H.S. of (4.7) in terms of integrals only involving U_1 , instead of gradients of U_1 . This can be done by using differential identities to integrate by parts. We first estimate the integral in (4.7) with $\chi_{E_\beta(\lambda)}$ and f replaced by their smoothing $\chi_\varepsilon := P_\varepsilon(\chi_{E_\beta(\lambda)})$ and $f_\varepsilon := P_\varepsilon(f)$, respectively, and then take the limit $\varepsilon \rightarrow 0$.

Lemma 4.1. *Let $\varepsilon > 0$. Suppose that $f_\varepsilon := f * P_\varepsilon$ for some real-valued function $f \in L^1(\mathbb{R}^n)$, $\chi_\varepsilon := \chi * P_\varepsilon$ with $0 \leq \chi \leq 1$ and $1 - \chi \in L^1(\mathbb{R}^n)$, and*

$$(4.8) \quad u := f_\varepsilon *_\mu P_{t_\mu}, \quad v := \chi_\varepsilon *_\mu P_{t_\mu}.$$

Then, we have $u(x, 0) = f_\varepsilon(x)$, $v(x, 0) = \chi_\varepsilon(x)$ and

$$(4.9) \quad \int_{\mathbb{R}^n \times \mathbb{R}_+} |\nabla_\mu u|^2 \phi^2(v) t_\mu dt_\mu dx \lesssim \int_{\mathbb{R}^n} u^2(x, 0) \phi^2(v(x, 0)) dx + \int_{\mathbb{R}^n \times \mathbb{R}_+} u^2 \Phi_1^2(v) |\nabla_\mu v|^2 t_\mu dt_\mu dx,$$

where ϕ satisfies (4.6), Φ_1 is an increasing $\mathcal{C}^2[0, 1]$ function satisfying $\Phi_1 = 0$ in a neighborhood of $[0, C_1]$ and, for every $r \in [0, 1]$,

$$(4.10) \quad \Phi_1(r) \geq |\phi(r)| + |\phi'(r)| + |\phi''(r)|.$$

Proof. By direct differentiation, we see that

$$(4.11) \quad \begin{aligned} (X_\mu^{(a)})^2 (u^2 \phi^2(v)) &= 2u(X_\mu^{(a)})^2 u \cdot \phi^2(v) + 2|X_\mu^{(a)} u|^2 \phi^2(v) + 8u\phi(v)\phi'(v)X_\mu^{(a)} u \cdot X_\mu^{(a)} v \\ &\quad + 2u^2\phi(v)\phi'(v)(X_\mu^{(a)})^2 v + 2u^2(\phi'(v)^2 + \phi(v)\phi''(v))|X_\mu^{(a)} v|^2, \end{aligned}$$

for $a = 1, 2$, where $(X_\mu^{(1)}, X_\mu^{(2)})$ is the gradient ∇_μ in (2.9). Since u and v satisfy partial Laplacian equation

$$(4.12) \quad \Delta_\mu u = 0, \quad \Delta_\mu v = 0,$$

on $\mathbb{R}^n \times \mathbb{R}_+$, by Proposition 2.1 (2), the summation of (4.11) over $a = 1, 2$ gives us

$$(4.13) \quad \begin{aligned} |\nabla_\mu u|^2 \phi^2(v) &= \frac{1}{2} \Delta_\mu (u^2 \phi^2(v)) - 4u\phi(v)\phi'(v) \nabla_\mu u \cdot \nabla_\mu v - u^2(\phi'(v)^2 + \phi(v)\phi''(v)) |\nabla_\mu v|^2 \\ &=: T_1 + T_2 + T_3. \end{aligned}$$

In addition,

$$|T_2| \leq \frac{1}{10} |\nabla_\mu u|^2 \phi^2(v) + 20u^2 \phi'(v)^2 |\nabla_\mu v|^2 := T_{21} + T_{22}.$$

The integral $\int_{\mathbb{R}^n \times \mathbb{R}_+} T_{21}(x, t_\mu) t_\mu dt_\mu dx$ can be absorbed by the integral of the L.H.S in (4.13), while

$$T_{22} + |T_3| \leq 40u^2 \Phi_1^2(v) |\nabla_\mu v|^2.$$

Then

$$\begin{aligned} \int_{\mathbb{R}^n \times \mathbb{R}_+} |\nabla_\mu u|^2 \phi^2(v) t_\mu dt_\mu dx &\leq \frac{5}{9} \int_{\mathbb{R}^n \times \mathbb{R}_+} \Delta_\mu(u^2 \phi^2(v)) t_\mu dt_\mu dx \\ &\quad + \frac{400}{9} \int_{\mathbb{R}^n \times \mathbb{R}_+} u^2 \Phi_1^2(v) |\nabla_\mu v|^2 t_\mu dt_\mu dx. \end{aligned}$$

Since $U = u^2 \phi^2(v)$ satisfies the assumptions in Proposition 2.3 by Corollary 3.1, we can apply Proposition 2.3 to $U = u^2 \phi^2(v)$ to get

$$\int_{\mathbb{R}^n \times \mathbb{R}_+} \Delta_\mu(u^2 \phi^2(v)) t_\mu dt_\mu dx = \int_{\mathbb{R}^n} u^2(x, 0) \phi^2(v(x, 0)) dx.$$

The estimate follows. $\square$

If we apply Lemma 4.1 to $u = U_1$ in (4.5) with f replaced by f_ε , there appears a term like

$$(4.14) \quad \int_{\mathbb{R}^n \times \mathbb{R}_+} |\nabla_2 U_2|^2 \Phi_1^2(v) |\nabla_1 v|^2 t_2 dt_2 dx.$$

Here for $k = 1, \dots, m$, we define functions

$$(4.15) \quad U_k(x, \mathbf{t}) := f_\varepsilon *_{t_1} P_{t_1} * \dots *_{t_k} P_{t_k} *_{t_{k+1}} \widetilde{\nabla}_{k+1} P_{t_{k+1}} * \dots *_{t_m} \widetilde{\nabla}_m P_{t_m}(x),$$

on $\mathbb{R}^n \times (\mathbb{R}_+)^m$, which satisfy

$$(4.16) \quad \nabla_k U_k(x, \mathbf{t}) = U_{k-1}(x, \mathbf{t}),$$

by Proposition 2.1 (2). So we need the following estimate further to estimate the term (4.14).

Lemma 4.2. *Keep the hypotheses and the notation of Lemma 4.1. Let*

$$u(x, t_\mu, t_\nu) = f_\varepsilon *_{t_\mu} P_{t_\mu} *_{t_\nu} P_{t_\nu}, \quad v(x, t_\mu, t_\nu) = \chi_\varepsilon *_{t_\mu} P_{t_\mu} *_{t_\nu} P_{t_\nu},$$

for $\mu \neq \nu$. Then for fixed t_μ , we have

$$(4.17) \quad \begin{aligned} \int_{\mathbb{R}^n \times \mathbb{R}_+} |\nabla_\nu u|^2 \Phi_1^2(v) |\nabla_\mu v|^2 \Big|_{(x, t_\mu, t_\nu)} t_\nu dt_\nu dx &\lesssim \int_{\mathbb{R}^n} u^2 \Phi_1^2(v) |\nabla_\mu v|^2 \Big|_{(x, t_\mu, 0)} dx \\ &\quad + \int_{\mathbb{R}^n \times \mathbb{R}_+} u^2 \Phi_2^2(v) \Sigma_{\mu\nu}(v) \Big|_{(x, t_\mu, t_\nu)} t_\nu dt_\nu dx, \end{aligned}$$

where

$$\Sigma_{\mu\nu}(v) := |\nabla_\mu v|^2 |\nabla_\nu v|^2 + |\nabla_\mu \nabla_\nu v|^2,$$

and Φ_2 is an increasing $\mathcal{C}^2[0, 1]$ function satisfying $\Phi_2 = 0$ in a neighborhood of $[0, C_1]$ and, for every $r \in [0, 1]$,

$$\Phi_2(r) \geq |\Phi_1(r)| + |\Phi_1'(r)| + |\Phi_1''(r)|.$$

Proof. By direct differentiation similar to (4.13), we have the following identity:

$$\begin{aligned}
|\nabla_\nu u|^2 \Phi_1^2(v) |\nabla_\mu v|^2 &= \frac{1}{2} \Delta_\nu \left(u^2 \Phi_1^2(v) |\nabla_\mu v|^2 \right) \\
&\quad - 4u \Phi_1(v) \Phi_1'(v) \nabla_\nu u \cdot \nabla_\nu v |\nabla_\mu v|^2 - 4u \Phi_1^2(v) \nabla_\nu u \cdot (\nabla_\mu v \cdot \nabla_\nu \nabla_\mu v) \\
&\quad - u^2 (\Phi_1'(v)^2 + \Phi_1(v) \Phi_1''(v)) |\nabla_\nu v|^2 |\nabla_\mu v|^2 \\
&\quad - 4u^2 \Phi_1(v) \Phi_1'(v) \nabla_\nu v \cdot (\nabla_\mu v \cdot \nabla_\nu \nabla_\mu v) - u^2 \Phi_1^2(v) |\nabla_\nu \nabla_\mu v|^2 \\
&=: \tilde{T}_1 + \tilde{T}_2 + \tilde{T}_3 + \tilde{T}_4 + \tilde{T}_5 + \tilde{T}_6,
\end{aligned} \tag{4.18}$$

where $\nabla_\nu u \cdot (\nabla_\mu v \cdot \nabla_\nu \nabla_\mu v) := \sum_{a,b=1,2} X_\nu^{(a)} u \cdot X_\mu^{(b)} v \cdot X_\nu^{(a)} X_\mu^{(b)} v$. Since $U = u^2 \Phi_1^2(v) |\nabla_\mu v|^2$ satisfies assumptions in Proposition 2.3 by the following Lemma 4.4, we can apply Proposition 2.3 to get

$$\int_{\mathbb{R}^n \times \mathbb{R}_+} \Delta_\nu \left(u^2 \Phi_1^2(v) |\nabla_\mu v|^2 \right) t_\nu dt_\nu dx = \int_{\mathbb{R}^n} u^2 \Phi_1^2(v) |\nabla_\mu v|^2 \Big|_{(x, t_\mu, 0)} dx.$$

Apply the Cauchy–Schwarz inequality to get

$$\left| \int_{\mathbb{R}^n \times \mathbb{R}_+} \tilde{T}_2 t_\nu dt_\nu dx \right| \leq \frac{1}{10} \int_{\mathbb{R}^n \times \mathbb{R}_+} |\nabla_\nu u|^2 \Phi_1^2(v) |\nabla_\mu v|^2 t_\nu dt_\nu dx + C \int_{\mathbb{R}^n \times \mathbb{R}_+} u^2 \Phi_2^2(v) |\nabla_\nu v|^2 |\nabla_\mu v|^2 t_\nu dt_\nu dx,$$

by definition of Φ_2 , and similarly

$$\left| \int_{\mathbb{R}^n \times \mathbb{R}_+} \tilde{T}_3 t_\nu dt_\nu dx \right| \leq \frac{1}{10} \int_{\mathbb{R}^n \times \mathbb{R}_+} |\nabla_\nu u|^2 \Phi_1^2(v) |\nabla_\mu v|^2 t_\nu dt_\nu dx + C \int_{\mathbb{R}^n \times \mathbb{R}_+} u^2 \Phi_1^2(v) |\nabla_\nu \nabla_\mu v|^2 t_\nu dt_\nu dx.$$

The first terms in the R. H. S. above are absorbed by the integral of the L.H.S. of (4.18). The integrals of $\tilde{T}_4, \tilde{T}_5, \tilde{T}_6$ are directly controlled by the second terms in the R. H. S. of (4.17). The lemma is proved. $\square$

4.3. Estimates in the general case. If we apply Lemma 4.2 to the term (4.14), there will appear a term like

$$\int_{\mathbb{R}^n \times \mathbb{R}_+} |\nabla_3 U_3|^2 \Phi_2^2(v) \Sigma_{12}(v) t_3 dt_3 dx. \tag{4.19}$$

Thus we need the following general estimates inductively. For a subset j of $\mathfrak{m}$, denote

$$\Sigma_j(v) := \sum_{j_1, \dots, j_l} |\nabla_{j_1} v|^2 \cdots |\nabla_{j_l} v|^2, \tag{4.20}$$

where the summation is taken over all partitions $j_1, \dots, j_l$ of j , i.e. j is the disjoint union of $j_1, \dots, j_l$.

Lemma 4.3. *Keep the hypotheses and the notation of Lemma 4.1. For a subset j of $\{1, 2, \dots, k-1\}$ with $a = |j|$, let*

$$u(x, \mathbf{t}_j, t_k) = f_\varepsilon *_{\mathbf{j}} P_{\mathbf{t}_j} *_{\mathbf{k}} P_{t_k}(x), \quad v(x, \mathbf{t}_j, t_k) = \chi_\varepsilon *_{\mathbf{j}} P_{\mathbf{t}_j} *_{\mathbf{k}} P_{t_k}(x). \tag{4.21}$$

Then for fixed $\mathbf{t}_j \in (\mathbb{R}_+)^a$, we have

$$\begin{aligned}
\int_{\mathbb{R}^n \times \mathbb{R}_+} |\nabla_k u|^2 \Phi_a^2(v) \Sigma_j(v) \Big|_{(x, \mathbf{t}_j, t_k)} t_k dt_k dx &\lesssim \int_{\mathbb{R}^n} u^2 \Phi_a^2(v) \Sigma_j(v) \Big|_{(x, \mathbf{t}_j, 0)} dx \\
&\quad + \int_{\mathbb{R}^n \times \mathbb{R}_+} u^2 \Phi_{a+1}(v)^2 \Sigma_{j \cup \{k\}}(v) \Big|_{(x, \mathbf{t}_j, t_k)} t_k dt_k dx,
\end{aligned} \tag{4.22}$$

where Φ_{a+1} is an increasing $\mathcal{C}^2[0, 1]$ function satisfying $\Phi_{a+1} = 0$ in a neighborhood of $[0, C_1]$ and, for every $r \in [0, 1]$, $\Phi_{a+1}(r) \geq |\Phi_a(r)| + |\Phi'_a(r)| + |\Phi''_a(r)|$. In particular, when $j = \emptyset$, $\Phi_0 = \phi$ and $\Sigma_\emptyset(v) = 1$.

Proof. Note that $\frac{1}{2}\Delta_k(u^2) = |\nabla_k u|^2$ and

$$\begin{aligned} \frac{1}{2}\Delta_k(\Phi_a^2(v)) &= (\Phi_a(v)\Phi_a''(v) + \Phi_a'(v)^2)|\nabla_k v|^2, \\ \frac{1}{2}\Delta_k(|\nabla_{j_\alpha} v|^2) &= \nabla_{j_\alpha} v \cdot \Delta_k \nabla_{j_\alpha} v + |\nabla_k \nabla_{j_\alpha} v|^2 = |\nabla_k \nabla_{j_\alpha} v|^2, \end{aligned}$$

by $\Delta_k u = 0 = \Delta_k v$ and $\Delta_k \nabla_{j_\alpha} v = \nabla_{j_\alpha} \Delta_k v = 0$ for any j_α and k . It follows from direct differentiation that

$$(4.23) \quad |\nabla_k u|^2 \Phi_a^2(v) \prod_{\beta} |\nabla_{j_\beta} v|^2 = \frac{1}{2} \Delta_k \left(u^2 \Phi_a^2(v) \prod_{\beta} |\nabla_{j_\beta} v|^2 \right) + \check{T}_2 + \check{T}_3 =: \check{T}_1 + \check{T}_2 + \check{T}_3,$$

where

$$(4.24) \quad \begin{aligned} \check{T}_2 &:= -4u\Phi_a(v)\Phi_a'(v)\nabla_k u \cdot \nabla_k v \prod_{\beta} |\nabla_{j_\beta} v|^2 \\ &\quad - 4u\Phi_a^2(v) \sum_{\alpha} \nabla_k u \cdot (\nabla_k \nabla_{j_\alpha} v \cdot \nabla_{j_\alpha} v) \prod_{\beta \neq \alpha} |\nabla_{j_\beta} v|^2 =: \check{T}_{21} + \check{T}_{22}, \end{aligned}$$

are terms involving first-order derivatives of u , and

$$(4.25) \quad \begin{aligned} \check{T}_3 &:= -u^2(\Phi_a(v)\Phi_a''(v) + \Phi_a'(v)^2)|\nabla_k v|^2 \prod_{\beta} |\nabla_{j_\beta} v|^2 \\ &\quad - 4u^2\Phi_a(v)\Phi_a'(v) \sum_{\alpha} \nabla_k v \cdot (\nabla_k \nabla_{j_\alpha} v \cdot \nabla_{j_\alpha} v) \prod_{\beta \neq \alpha} |\nabla_{j_\beta} v|^2 \\ &\quad - u^2\Phi_a^2(v) \sum_{\alpha} |\nabla_k \nabla_{j_\alpha} v|^2 \prod_{\beta \neq \alpha} |\nabla_{j_\beta} v|^2 \\ &\quad - 4u^2\Phi_a^2(v) \sum_{\alpha \neq \gamma} (\nabla_k \nabla_{j_\alpha} v \cdot \nabla_{j_\alpha} v) \cdot (\nabla_k \nabla_{j_\gamma} v \cdot \nabla_{j_\gamma} v) \prod_{\beta \neq \alpha, \gamma} |\nabla_{j_\beta} v|^2 \\ &=: \check{T}_{31} + \check{T}_{32} + \check{T}_{33} + \check{T}_{34}, \end{aligned}$$

are terms only involving derivatives of v .

Since $U = u^2\Phi_a^2(v) \prod_{\beta} |\nabla_{j_\beta} v|^2$ satisfies assumptions in Proposition 2.3 by the ensuing Lemma 4.4, we can apply Proposition 2.3 to get the integral of $\check{T}_1$ controlled by the first term in R.H.S. of (4.22). Similarly, by applying Cauchy–Schwarz inequality, we have

$$(4.26) \quad \begin{aligned} \int_{\mathbb{R}^n \times \mathbb{R}_+} |\check{T}_{21}(x, \mathbf{t}_j, t_k)| t_k dt_k dx &\leq \frac{1}{10} \int_{\mathbb{R}^n \times \mathbb{R}_+} |\nabla_k u|^2 \Phi_a^2(v) \prod_{\beta} |\nabla_{j_\beta} v|^2 t_k dt_k dx \\ &\quad + C \int_{\mathbb{R}^n \times \mathbb{R}_+} u^2 \Phi_{a+1}^2(v) |\nabla_k v|^2 \prod_{\beta} |\nabla_{j_\beta} v|^2 t_k dt_k dx, \\ \int_{\mathbb{R}^n \times \mathbb{R}_+} |\check{T}_{22}(x, \mathbf{t}_j, t_k)| t_k dt_k dx &\leq \frac{1}{10} \int_{\mathbb{R}^n \times \mathbb{R}_+} |\nabla_k u|^2 \Phi_a^2(v) \prod_{\beta} |\nabla_{j_\beta} v|^2 t_k dt_k dx \\ &\quad + C \int_{\mathbb{R}^n \times \mathbb{R}_+} u^2 \Phi_a^2(v) \sum_{\alpha} |\nabla_k \nabla_{j_\alpha} v|^2 \prod_{\beta \neq \alpha} |\nabla_{j_\beta} v|^2 t_k dt_k dx. \end{aligned}$$

The first terms in the R.H.S. of (4.26) can be absorbed by the integral of the L.H.S. of (4.23). The second terms in the R.H.S. of (4.26) can be controlled by the second terms in the R.H.S. of (4.22), by definition of $\Sigma_{j \cup \{k\}}$.

The integrals of $\check{T}_{3j}$'s are controlled directly by the second term in R.H.S. of (4.22) by applying the Cauchy–Schwarz inequality. The estimate (4.22) follows. $\square$

Lemma 4.4. *Let $\varepsilon > 0$ and let u, v be given as in Lemma 4.3. Then, for each partition $j_1, \dots, j_l$ of j , $U = u^2 \Phi_a^2(v) \prod_{\beta} |\nabla_{j_\beta} v|^2$ satisfies the conditions in Proposition 2.3.*

Proof. We claim the following facts:

$$(4.27) \quad \begin{aligned} &u^2 \text{ and } |\nabla_{j_\beta} v|^2 \text{ satisfy the conditions in Proposition 2.3,} \\ &v \text{ and all its derivatives appearing below are uniformly bounded.} \end{aligned}$$

This is because we can apply Corollary 3.1 for $\mu = k$ to $u = f_\varepsilon * j P_{t_j} *_{\varepsilon} P_{t_k} = (f * j P_{t_j})_\varepsilon *_{\varepsilon} P_{t_k}$ with f replaced by $f * j P_{t_j}$, which belongs to $L^1(\mathbb{R}^n)$ for fixed $t_j \in (\mathbb{R}_+)^a$ by Proposition 3.4. Note that

$$\nabla_{j_\beta} v(x, t_j, t_k) = \nabla_{j_\beta} (\chi_\varepsilon * j P_{t_j} *_{\varepsilon} P_{t_k}) = (\nabla_{j_\beta} (\chi * j P_{t_j}))_\varepsilon *_{\varepsilon} P_{t_k},$$

where the last identity follows from the commutativity of differentiation and convolution, together with the semigroup property of Poisson integral. On the other hand, since $1 - \chi \in L^1(\mathbb{R}^n)$, we must have $\nabla_{j_\beta} (\chi * j P_{t_j}) = \nabla_{j_\beta} ((\chi - 1) * j P_{t_j}) \in L^1(\mathbb{R}^n)$ for fixed $t_j \in (\mathbb{R}_+)^a$ by Proposition 3.4 again. So we can apply Corollary 3.1 to obtain the claim for $|\nabla_{j_\beta} v|^2$. By the same reason, $t_k^b \partial_{t_k}^b \nabla_{j_\beta} v$ is uniformly bounded.

Now for vector fields $X = X_k, \partial_{t_k}, t_k \partial_{t_k}$, we have

$$(4.28) \quad \begin{aligned} XU &= (X(u^2) \Phi_a^2(v) + 2u^2 \Phi_a(v) \Phi'_a(v) Xv) \prod_{\beta} |\nabla_{j_\beta} v|^2 \\ &\quad + \sum_{\alpha} u^2 \Phi_a^2(v) X(|\nabla_{j_\alpha} v|^2) \prod_{\beta \neq \alpha} |\nabla_{j_\beta} v|^2. \end{aligned}$$

Then, $\nabla_k U(\cdot, t_k) \in L^1(\mathbb{R}^n)$ because Φ, Φ' are bounded, and each term above contains at least one $L^1 \cap L^\infty$ factor and all remaining factors are uniformly bounded by (4.27).

We also have $X_k U(x^\perp + s e_k, t_k) \rightarrow 0$ as $s \rightarrow \pm\infty$ for almost all $x^\perp \in e_k^\perp$, because $X_k(u^2), X_k v$ and $X_k(|\nabla_{j_\alpha} v|^2)$ have this property. Similarly, $\nabla_k^a U(\cdot, t_k) \in L^1(\mathbb{R}^n)$ for $a = 0, 2$.

$\lim_{t_k \rightarrow +\infty} \|t_k^b \partial_{t_k}^b U(\cdot, t_k)\|_{L^1(\mathbb{R}^n)} = 0$ for $b = 0, 1$, because $u^2, t_k \partial_{t_k}(u^2) \rightarrow 0$ in $L^1(\mathbb{R}^n)$ as $t_k \rightarrow +\infty$ by (4.27), and other factors in (4.28) are all bounded by a constant independent of x, t_k . By (4.27) again,

$$u^2(\cdot, t_k) \rightarrow u^2(\cdot, 0), \quad |\nabla_{j_\beta} v|^2(\cdot, t_k) \rightarrow |\nabla_{j_\beta} v|^2(\cdot, 0),$$

in $L^1(\mathbb{R}^n)$ as $t_k \rightarrow 0$, and $\Phi_a^2(v)(\cdot, t_k) \rightarrow \Phi_a^2(v)(\cdot, 0)$ a.e., which is bounded. Consequently, $U(\cdot, t_k) \rightarrow U(\cdot, 0)$ in $L^1(\mathbb{R}^n)$ as $t_k \rightarrow 0$. Similarly, $t_k \partial_{t_k} U(\cdot, t_k) \rightarrow 0$ in $L^1(\mathbb{R}^n)$ as $t_k \rightarrow 0$. $\square$

4.4. Proof of the Fefferman–Stein type good- λ inequality.

Proposition 4.2. *Let $\varepsilon > 0$. Suppose that $f_\varepsilon := f * P_\varepsilon$ for some complex-valued function $f \in L^1(\mathbb{R}^n)$, $\chi_\varepsilon := \chi_E * P_\varepsilon$ with E^c having bounded measure, and $v(x, t) = \chi_\varepsilon * P_t$. Then, we have the estimate*

$$(4.29) \quad I = \int_{\mathbb{R}^n \times (\mathbb{R}_+)^m} |\nabla_m(f_\varepsilon * P_t)|^2 \phi^2(v) t dt dx \lesssim \sum_j \int_{\mathbb{R}^n \times (\mathbb{R}_+)^{|j|}} |P_{t_j}(f_\varepsilon)|^2 \Phi_{|j|}^2(v) \Sigma_j(v) \Big|_{(x, 0, j^c, t_j)} t_j dt_j dx,$$

where the summation is taken over subsets $\mathbf{j}$ of $\mathbf{m}$, $\Sigma_{\mathbf{j}}(v)$ is given by (4.20), $\mathbf{j}^c := \mathbf{m} \setminus \mathbf{j}$, and $\Phi_{|\mathbf{j}|}$ is taken as in Lemma 4.3. In particular, when $\mathbf{j} = \emptyset$, $\Phi_0 = \phi$, $\Sigma_{\emptyset}(v) = 1$ and $P_{\mathbf{t}_{\emptyset}}(f_{\varepsilon}) = f_{\varepsilon}$.

Proof. Applying estimate (4.9) in Lemma 4.1 to the real and imaginary parts of $U_1(x, t_1, \mathbf{t}_{\mathbf{m}_2})$ given by (4.15)-(4.16) for fixed $\mathbf{t}_{\mathbf{m}_2}$, where $\mathbf{m}_2$ is defined in (2.11), we get the integral I in (4.29) can be estimated as

$$\begin{aligned}
 (4.30) \quad I &= \int_{(\mathbb{R}_+)^{m-1}} \int_{\mathbb{R}^n \times \mathbb{R}_+} |\nabla_1 U_1(x, t_1, \mathbf{t}_{\mathbf{m}_2})|^2 \phi^2(v) t_1 dt_1 dx \mathbf{t}_{\mathbf{m}_2} d\mathbf{t}_{\mathbf{m}_2} \\
 &\lesssim \int_{(\mathbb{R}_+)^{m-1}} \left(\int_{\mathbb{R}^n \times \mathbb{R}_+} |U_1|^2 \Phi_1^2(v) |\nabla_1 v|^2 \right)_{(x, \mathbf{t})} t_1 dt_1 dx + \int_{\mathbb{R}^n} |U_1|^2 \phi^2(v) \Big|_{(x, 0, \mathbf{t}_{\mathbf{m}_2})} dx \Big) \mathbf{t}_{\mathbf{m}_2} d\mathbf{t}_{\mathbf{m}_2} \\
 &= \int_{\mathbb{R}^n \times (\mathbb{R}_+)^m} |U_1|^2 \Phi_1^2(v) |\nabla_1 v|^2 \Big|_{(x, \mathbf{t})} \mathbf{t} d\mathbf{t} dx + \int_{\mathbb{R}^n \times (\mathbb{R}_+)^{m-1}} |U_1|^2 \phi^2(v) \Big|_{(x, 0, \mathbf{t}_{\mathbf{m}_2})} \mathbf{t}_{\mathbf{m}_2} d\mathbf{t}_{\mathbf{m}_2} dx.
 \end{aligned}$$

We claim that

$$(4.31) \quad I \lesssim \sum_{a=0}^{k-1} \sum_{|\mathbf{j}|=a} S_{\mathbf{j}}^{(k)},$$

where the summation is taken over all subsets $\mathbf{j}$ of $\{1, 2, \dots, k-1\}$, and

$$(4.32) \quad S_{\mathbf{j}}^{(k)} := \int_{\mathbb{R}^n \times \mathbb{R}_+^{m-k+1+|\mathbf{j}|}} |U_{k-1}|^2 \Phi_{|\mathbf{j}|}^2(v) \Sigma_{\mathbf{j}}(v) \Big|_{(x, \mathbf{0}_{\mathbf{j}_k^c}, \mathbf{t}_{\mathbf{j} \cup \mathbf{m}_k})} \mathbf{t}_{\mathbf{j} \cup \mathbf{m}_k} d\mathbf{t}_{\mathbf{j} \cup \mathbf{m}_k} dx,$$

$k = 2, \dots, m+1$. Here $\Sigma_{\mathbf{j}}(v) = 1$ if $\mathbf{j} = \emptyset$, U_k is given by (4.15), $U_{k-1} = \nabla_k U_k$, $\mathbf{m}_k = \{k, \dots, m\}$, and

$$\mathbf{j}_k^c := \{1, 2, \dots, k-1\} \setminus \mathbf{j}.$$

Let us prove the claim (4.31) inductively. (4.30) implies that the claim (4.31) already holds for $k = 2$. Assume that (4.31) holds for a positive integer $k \leq m$. Let

$$\tilde{u}(x, \mathbf{t}_{\mathbf{j}}, t_k) = \tilde{f}_{\varepsilon} *_{\mathbf{j}} P_{\mathbf{t}_{\mathbf{j}}} *_{t_k} P_{t_k}(x), \quad \tilde{v}(x, \mathbf{t}_{\mathbf{j}}, t_k) = \tilde{\chi}_{\varepsilon} *_{\mathbf{j}} P_{\mathbf{t}_{\mathbf{j}}} *_{t_k} P_{t_k}(x),$$

for fixed $\mathbf{t}_{\mathbf{m}_{k+1}}$, where

$$\tilde{f}_{\varepsilon}(x) = f_{\varepsilon} *_{k+1} \tilde{\nabla}_{k+1} P_{t_{k+1}} * \dots *_{\mathbf{m}} \tilde{\nabla}_{\mathbf{m}} P_{t_{\mathbf{m}}}(x), \quad \tilde{\chi}_{\varepsilon}(x) = \chi_{\varepsilon} *_{k+1} P_{t_{k+1}} * \dots *_{\mathbf{m}} P_{t_{\mathbf{m}}}(x).$$

Note that $\tilde{f}_{\varepsilon} \in L^1(\mathbb{R}^n)$ by applying (3.16) repeatedly, and similarly, $0 \leq \tilde{\chi}_{\varepsilon} \leq 1$ and $1 - \tilde{\chi}_{\varepsilon} \in L^1(\mathbb{R}^n)$. Moreover, their L^1 -norms are independent of $\mathbf{t}$ since $\|P_t\|_{L^1(\mathbb{R})} = 1$. Then, by definition,

$$U_k|_{(x, \mathbf{0}_{\mathbf{j}_k^c}, \mathbf{t}_{\mathbf{j} \cup \mathbf{m}_k})} = \tilde{u}(x, \mathbf{t}_{\mathbf{j}}, t_k), \quad v|_{(x, \mathbf{0}_{\mathbf{j}_k^c}, \mathbf{t}_{\mathbf{j} \cup \mathbf{m}_k})} = \tilde{v}(x, \mathbf{t}_{\mathbf{j}}, t_k)$$

for fixed $\mathbf{t}_{\mathbf{m}_{k+1}}$. Hence, for each $\mathbf{t}_{\mathbf{j}}$ and $\mathbf{t}_{\mathbf{m}_{k+1}}$, the hypotheses in Lemma 4.3 are satisfied. So we can apply the estimate (4.22) in Lemma 4.3 to $u = \tilde{u}$ and $v = \tilde{v}$ for fixed $\mathbf{t}_{\mathbf{j}}$ and $\mathbf{t}_{\mathbf{m}_{k+1}}$ to get

$$\begin{aligned} S_{\mathbf{j}}^{(k)} &= \int_{(\mathbb{R}_+)^{m-k+|\mathbf{j}|}} \left(\int_{\mathbb{R}^n \times \mathbb{R}_+} |\nabla_k \tilde{u}|^2 \Phi_{|\mathbf{j}|}^2(\tilde{v}) \Sigma_{\mathbf{j}}(\tilde{v}) \right)_{(x, \mathbf{t}_{\mathbf{j}}, t_k)} t_k dt_k dx \Bigg) \mathbf{t}_{\mathbf{j} \cup \mathbf{m}_{k+1}} d\mathbf{t}_{\mathbf{j} \cup \mathbf{m}_{k+1}} \\ &\lesssim \int_{(\mathbb{R}_+)^{m-k+|\mathbf{j}|}} \left(\int_{\mathbb{R}^n} |\tilde{u}|^2 \Phi_{|\mathbf{j}|}^2(\tilde{v}) \Sigma_{\mathbf{j}}(\tilde{v}) \right)_{(x, \mathbf{t}_{\mathbf{j}}, 0)} dx \\ &\quad + \int_{\mathbb{R}^n \times \mathbb{R}_+} |\tilde{u}|^2 \Phi_{|\mathbf{j}|+1}^2(\tilde{v}) \Sigma_{\mathbf{j} \cup \{\mathbf{k}\}}(\tilde{v}) \Bigg)_{(x, \mathbf{t}_{\mathbf{j}}, t_k)} t_k dt_k dx \Bigg) \mathbf{t}_{\mathbf{j} \cup \mathbf{m}_{k+1}} d\mathbf{t}_{\mathbf{j} \cup \mathbf{m}_{k+1}} \end{aligned}$$

Thus,

$$\begin{aligned} S_{\mathbf{j}}^{(k)} &= \int_{(\mathbb{R}_+)^{m-k+|\mathbf{j}|}} |U_k|^2 \Phi_{|\mathbf{j}|}^2(v) \Sigma_{\mathbf{j}}(v) \Bigg)_{(x, \mathbf{0}_{\mathbf{j}_k^c}, \mathbf{t}_{\mathbf{j} \cup \mathbf{m}_{k+1}})} \mathbf{t}_{\mathbf{j} \cup \mathbf{m}_{k+1}} d\mathbf{t}_{\mathbf{j} \cup \mathbf{m}_{k+1}} dx \\ &\quad + \int_{\mathbb{R}^n \times (\mathbb{R}_+)^{m-k+|\mathbf{j}|+1}} |U_k|^2 \Phi_{|\mathbf{j}|+1}^2(v) \Sigma_{\mathbf{j} \cup \{\mathbf{k}\}}(v) \Bigg)_{(x, \mathbf{0}_{\mathbf{j}_k^c}, \mathbf{t}_{\mathbf{j} \cup \mathbf{m}_k})} \mathbf{t}_{\mathbf{j} \cup \mathbf{m}_k} d\mathbf{t}_{\mathbf{j} \cup \mathbf{m}_k} dx \\ &= S_{\mathbf{j}}^{(k+1)} + S_{\mathbf{j} \cup \{\mathbf{k}\}}^{(k+1)}. \end{aligned}$$

It follows that

$$\sum_{a=0}^{k-1} \sum_{|\mathbf{j}|=a} S_{\mathbf{j}}^{(k)} \lesssim \sum_{a=0}^k \sum_{|\mathbf{l}|=a} S_{\mathbf{l}}^{(k+1)}$$

by definition, where the summation in the right hand side is taken over subsets $\mathbf{l}$ of $\{1, 2, \dots, k\}$. The R.H.S. of the estimate (4.29) is exactly (4.31) for $k = m + 1$, since $U_m|_{(x, \mathbf{0}_{\mathbf{j}_k^c}, \mathbf{t}_{\mathbf{j}})} = P_{\mathbf{t}_{\mathbf{j}}}(f_\varepsilon)$ and $\mathbf{m}_{m+1} = \emptyset$ by definition (4.15). $\square$

The integrals in Proposition 4.2 can be further estimated as follows.

Lemma 4.5. *Assume as in Proposition 4.2. Then for a nonempty subset $\mathbf{j}$ of $\mathbf{m}$, $E = E_\beta(\lambda)$, and $\lambda > 0$, we have*

$$(4.33) \quad \int_{\mathbb{R}^n \times (\mathbb{R}_+)^{|\mathbf{j}|}} |P_{\mathbf{t}_{\mathbf{j}}}(f_\varepsilon)|^2 \Phi_{|\mathbf{j}|}^2(v)^2 \Sigma_{\mathbf{j}}(v) \Bigg)_{(x, \mathbf{0}_{\mathbf{j}_k^c}, \mathbf{t}_{\mathbf{j}})} \mathbf{t}_{\mathbf{j}} d\mathbf{t}_{\mathbf{j}} dx \lesssim \lambda^2 |E_\beta(\lambda)^c|.$$

Proof. For $\Phi_{|\mathbf{j}|}(P_{\mathbf{t}_{\mathbf{j}}}(\chi_\varepsilon))(x) \neq 0$, we have

$$P_{\mathbf{t}_{\mathbf{j}}+\varepsilon}(\chi)(x) = P_{\mathbf{t}_{\mathbf{j}}}(\chi_\varepsilon)(x) > C_1$$

by definition of $\Phi_{|\mathbf{j}|}$. Here, by abuse of notation, we denote $(\mathbf{0}_{\mathbf{j}_k^c}, \mathbf{t}_{\mathbf{j}})$ by $\mathbf{t}_{\mathbf{j}}$ briefly. Consequently, $(x, \mathbf{t}_{\mathbf{j}}+\varepsilon) \in \widetilde{W}_\beta$ by Proposition 4.1. Hence, there exists $x' \in E_\beta(\lambda)$ such that $(x, \mathbf{t}_{\mathbf{j}}+\varepsilon) \in \Gamma_\beta(x')$, and so

$$|P_{\mathbf{t}_{\mathbf{j}}}(f_\varepsilon)(x)| = |P_{\mathbf{t}_{\mathbf{j}}+\varepsilon}(f)(x)| \leq N^\beta(f)(x') \leq \lambda.$$

Therefore,

$$\int_{\mathbb{R}^n \times (\mathbb{R}_+)^{|\mathbf{j}|}} |P_{\mathbf{t}_{\mathbf{j}}}(f_\varepsilon)|^2 \Phi_{|\mathbf{j}|}^2(v) \Sigma_{\mathbf{j}}(v) \Bigg)_{(x, \mathbf{t}_{\mathbf{j}})} \mathbf{t}_{\mathbf{j}} d\mathbf{t}_{\mathbf{j}} dx \leq C \sum_{\mathbf{j}_1, \dots, \mathbf{j}_l} \lambda^2 \int_{\mathbb{R}^n \times (\mathbb{R}_+)^{|\mathbf{j}|}} |\nabla_{\mathbf{j}_1} v|^2 \cdots |\nabla_{\mathbf{j}_l} v|^2 \Bigg)_{(x, \mathbf{t}_{\mathbf{j}})} \mathbf{t}_{\mathbf{j}} d\mathbf{t}_{\mathbf{j}} dx$$

where $C = \max\{\Phi_a^2(r) : r \in [0, 1], a = 1, \dots, m\}$, the summation is taken over partitions $j_1, \dots, j_l$ of j , and

$$v(x, \mathbf{t}_j) = P_{\mathbf{t}_j}(\chi_\varepsilon)(x) = P_{\mathbf{t}_j + \varepsilon}(\chi)(x).$$

Recall that $|P_{t_\mu}(h)| \lesssim M_\mu(h)$, $|t_\mu \nabla_\mu P_{t_\mu}(h)| \lesssim M_\mu(h)$ for $h \in L^1(\mathbb{R})$. This together with

$$\nabla_{j_\alpha} v(x, \mathbf{t}_j) = \chi_\varepsilon *_{j_1} P_{\mathbf{t}_{j_1}} \cdots *_{j_\alpha} \widetilde{\nabla}_{j_\alpha} P_{\mathbf{t}_{j_\alpha}} * \cdots *_{j_l} P_{\mathbf{t}_{j_l}} = \nabla_{j_\alpha} (\chi_\varepsilon *_{j_\alpha} P_{\mathbf{t}_{j_\alpha}}) *_{\mathbf{t}_{j \setminus j_\alpha}} P_{\mathbf{t}_{j \setminus j_\alpha}},$$

by the commutativity (2.4) of partial convolutions and (3.1), implies

$$|\mathbf{t}_{j_\alpha} \nabla_{j_\alpha} v(x, \mathbf{t}_j)|^2 \lesssim |M_{j \setminus j_\alpha}(\mathbf{t}_{j_\alpha} \nabla_{j_\alpha} P_{\mathbf{t}_{j_\alpha}}(\chi_\varepsilon))(x)|^2,$$

where $j \setminus j_\alpha$ is the complement of j_α in j , and $M_{j \setminus j_\alpha} = M_{\mu_c} \circ \cdots \circ M_{\mu_1}$ is an iterated maximal function if $j \setminus j_\alpha = \{j_1, \dots, j_c\}$ ($c = |j| - |j_\alpha|$). For a subset $j = \{j_1, \dots, j_a\}$ of $\mathbf{m}$ ($a = |j|$), denote $\frac{d\mathbf{t}_j}{\mathbf{t}_j} := \frac{dt_{j_1} \cdots dt_{j_a}}{t_{j_1} \cdots t_{j_a}}$. Then,

$$\begin{aligned} \int_{\mathbb{R}^n \times (\mathbb{R}_+)^{|j|}} |\nabla_{j_1} v|^2 \cdots |\nabla_{j_l} v|^2 \mathbf{t}_j d\mathbf{t}_j dx &\lesssim \int_{\mathbb{R}^n \times (\mathbb{R}_+)^{|j|}} \prod_{\alpha=1}^l |M_{j \setminus j_\alpha}(\mathbf{t}_{j_\alpha} \nabla_{j_\alpha} P_{\mathbf{t}_{j_\alpha}}(\chi_\varepsilon))|^2 dx \frac{d\mathbf{t}_j}{\mathbf{t}_j} \\ &= \int_{\mathbb{R}^n} \prod_{\alpha=1}^l \int_{(\mathbb{R}_+)^{|j_\alpha|}} |M_{j \setminus j_\alpha}(\mathbf{t}_{j_\alpha} \nabla_{j_\alpha} P_{\mathbf{t}_{j_\alpha}}(\chi_\varepsilon))|^2 \frac{d\mathbf{t}_{j_\alpha}}{\mathbf{t}_{j_\alpha}} dx \\ &\leq \prod_{\alpha=1}^l \left| \int_{\mathbb{R}^n} \left(\int_{(\mathbb{R}_+)^{|j_\alpha|}} |M_{j \setminus j_\alpha}(\mathbf{t}_{j_\alpha} \nabla_{j_\alpha} P_{\mathbf{t}_{j_\alpha}}(1 - \chi_\varepsilon))|^2 \frac{d\mathbf{t}_{j_\alpha}}{\mathbf{t}_{j_\alpha}} \right)^l dx \right|^{\frac{1}{l}} \\ &\lesssim \prod_{\alpha=1}^l \left| \int_{\mathbb{R}^n} g_{j_\alpha}^{2l}(1 - \chi_\varepsilon) dx \right|^{\frac{1}{l}} \\ &\lesssim \|1 - \chi_\varepsilon\|_{L^{2l}(\mathbb{R}^n)}^{2l} = \|(1 - \chi)_\varepsilon\|_{L^{2l}(\mathbb{R}^n)}^{2l} \\ &\leq \|1 - \chi\|_{L^{2l}(\mathbb{R}^n)}^{2l} = |E_\beta(\lambda)^c|. \end{aligned}$$

by using Hölder's inequality, the vector-valued inequality for the Hardy-Littlewood maximal functions, L^{2l} boundedness of partial Littlewood-Paley g -function in Proposition 2.2, and Young's inequality. $\square$

Proof of Theorem 4.1. Recall that $E_\beta(\lambda) = \{x \in \mathbb{R}^n : N^\beta(f)(x) \leq \lambda\}$. Since $N^\beta(f) \in L^1(\mathbb{R}^n)$, we have $|E_\beta(\lambda)^c| < \infty$. For $\varepsilon > 0$, set

$$f_\varepsilon = f * P_\varepsilon, \quad \chi_\varepsilon = \chi * P_\varepsilon, \quad v_\varepsilon(x, \mathbf{t}) = \chi_\varepsilon * P_{\mathbf{t}}(x),$$

where $\chi = \chi_{E_\beta(\lambda)}$ and $\varepsilon = (\varepsilon, \dots, \varepsilon)$. Then $v_\varepsilon(x, \mathbf{0}) = \chi_\varepsilon(x)$. By Proposition 4.2 and Lemma 4.5,

$$(4.34) \quad \int_{\mathbb{R}^n \times (\mathbb{R}_+)^m} |\nabla_{\mathbf{m}}(f_\varepsilon * P_{\mathbf{t}})(x)|^2 \phi^2(v_\varepsilon(x, \mathbf{t})) \mathbf{t} d\mathbf{t} dx \lesssim \int_{\mathbb{R}^n} |f_\varepsilon(x)|^2 \phi^2(\chi_\varepsilon(x)) dx + \lambda^2 |E_\beta(\lambda)^c|.$$

We claim that

$$(4.35) \quad |f_\varepsilon(x)|^2 \phi^2(\chi_\varepsilon(x)) \leq \lambda N^\beta(f)(x).$$

It is trivial where $\phi(\chi_\varepsilon(x)) = 0$. Otherwise, since $\phi = 0$ on $[0, C_1]$, we have

$$\chi_\varepsilon(x) = P_\varepsilon(\chi_{E_\beta(\lambda)})(x) > C_1.$$

By Proposition 4.1(2), this implies $(x, \varepsilon) \in \widetilde{W}_\beta$. Hence there exists $x' \in E_\beta(\lambda)$ such that $(x, \varepsilon) \in \Gamma_\beta(x')$. Therefore

$$|f_\varepsilon(x)| = |f * P_\varepsilon(x)| \leq N^\beta(f)(x') \leq \lambda.$$

On the other hand, since $(x, \varepsilon) \in \Gamma_\beta(x)$, by the definition of the nontangential maximal function,

$$|f_\varepsilon(x)| = |f * P_\varepsilon(x)| \leq N^\beta(f)(x).$$

Thus, on the set where $\phi(\chi_\varepsilon(x)) \neq 0$, $|f_\varepsilon(x)|^2 \leq \lambda N^\beta(f)(x)$. Since $0 \leq \phi \leq 1$, (4.35) follows.

By the smoothness of $f * P_t(x)$ and $\chi * P_t(x)$ in Proposition 3.3 (c), we have

$$\begin{aligned} \nabla_m(f_\varepsilon * P_t)(x) &= \nabla_m(f * P_{t+\varepsilon})(x) \rightarrow \nabla_m(f * P_t)(x), \\ v_\varepsilon(x, t) &= \chi * P_{t+\varepsilon}(x) \rightarrow \chi * P_t(x) = v(x, t), \end{aligned}$$

for every $(x, t) \in \mathbb{R}^n \times (\mathbb{R}_+)^m$, as $\varepsilon \rightarrow 0$. Note that by Proposition 3.3, $\chi_\varepsilon(x) \rightarrow \chi(x)$ for a.e. $x \in \mathbb{R}^n$, and $f_\varepsilon \rightarrow f$ in $L^1(\mathbb{R}^n)$. Therefore, there exists a sequence $\varepsilon_j \rightarrow 0$ such that $f_{\varepsilon_j}(x) \rightarrow f(x)$ for a.e. $x \in \mathbb{R}^n$. Hence, by Fatou's lemma, and (4.34), we have

$$\begin{aligned} \int_{\mathbb{R}^n \times (\mathbb{R}_+)^m} |\nabla_m(f * P_t)(x)|^2 \phi^2(v(x, t)) t \, dt \, dx &\leq \liminf_{j \rightarrow \infty} \int_{\mathbb{R}^n \times (\mathbb{R}_+)^m} |\nabla_m(f_{\varepsilon_j} * P_t)(x)|^2 \phi^2(v_{\varepsilon_j}(x, t)) t \, dt \, dx \\ &\lesssim \liminf_{j \rightarrow \infty} \int_{\mathbb{R}^n} |f_{\varepsilon_j}(x)|^2 \phi^2(\chi_{\varepsilon_j}(x)) \, dx + \lambda^2 |E_\beta(\lambda)^c| \\ &= \int_{\mathbb{R}^n} |f(x)|^2 \phi^2(\chi(x)) \, dx + \lambda^2 |E_\beta(\lambda)^c| \\ &\leq \int_{E_\beta(\lambda)} N^\beta(f)(x)^2 \, dx + \lambda^2 |E_\beta(\lambda)^c|. \end{aligned}$$

Here we used the dominated convergence theorem with the controlling function given by (4.35), the fact that $|f| \leq N^\beta(f)$ a.e., and $\phi^2(\chi) = \chi_{E_\beta(\lambda)}$. Finally, substituting this estimate into (4.7), together with (4.1), we get

$$|\{x : S(f)(x) > \lambda\}| \lesssim |E_\beta(\lambda)^c| + \frac{1}{\lambda^2} \int_{E_\beta(\lambda)} N^\beta(f)^2(x) \, dx.$$

Since $E_\beta(\lambda)^c = \{x : N^\beta(f)(x) > \lambda\}$, this is exactly the inequality (1.14). The proof is complete. $\square$

5. $\mathfrak{H}$ -VALUED LITTLEWOOD-PALEY g FUNCTION AND S FUNCTION

5.1. The Plancherel-Pólya type inequality for $\mathfrak{H}$ -valued functions. Let $\mathfrak{H}$ be a separable Hilbert space. A $\mathfrak{H}$ -valued function is called *simple* if it can be written as $\sum_{n=1}^N \chi_{A_n} x_n$, where $x_n \in \mathfrak{H}$ and $A_n \subset \mathbb{R}$ measurable with $|A_n| < \infty$ for each $1 \leq n \leq N$. A $\mathfrak{H}$ -valued function f is *strongly measurable* if there exists a sequence $\{\psi_n\}$ of simple functions such that $\psi_n \rightarrow f$ almost everywhere on $\mathbb{R}$. A strongly measurable $\mathfrak{H}$ -valued function is said to be *Bochner integrable* if there exists a sequence of $\mathfrak{H}$ -valued simple functions $\{\psi_n\}$ satisfying $\lim_{n \rightarrow \infty} \int_{\mathbb{R}} \|f(x) - \psi_n(x)\|_{\mathfrak{H}} \, dx = 0$. The limit is defined as the Bochner integral of f , denoted by $\int_{\mathbb{R}} f(x) \, dx := \lim_{n \rightarrow \infty} \int_{\mathbb{R}} \psi_n(x) \, dx$. Let $\{h_1, h_2, \dots\}$ be a complete orthonormal basis of $\mathfrak{H}$. Then, we have the expansion

$$(5.1) \quad f(x) = \sum_{i=1}^{\infty} f_i(x) h_i, \quad f_i(x) = \langle f(x), h_i \rangle_{\mathfrak{H}}$$

which converges pointwise. f_i must be measurable by the approximation of simple functions.

For $1 \leq p < \infty$, the space $L^p(\mathbb{R}, \mathfrak{H})$ is defined as the linear space of all equivalence classes of strongly measurable functions f such that $\|f\|_{L^p(\mathbb{R}, \mathfrak{H})} := (\int_{\mathbb{R}} \|f(x)\|_{\mathfrak{H}}^p \, dx)^{\frac{1}{p}} < \infty$. They are Banach spaces (cf. [25]). In particular, $L^2(\mathbb{R}, \mathfrak{H})$ is a Hilbert space and $\|f\|_{L^2(\mathbb{R}, \mathfrak{H})}^2 = \sum_{i=1}^{\infty} \|f_i\|_{L^2(\mathbb{R})}^2$. For $f \in L^p(\mathbb{R}, \mathfrak{H})$, let $f_N := \sum_{i=1}^N f_i h_i$ for $N \in \mathbb{N}$. Then, $\|f_N - f\|_{L^p(\mathbb{R}, \mathfrak{H})} \rightarrow 0$. This follows from the dominated convergence theorem [25, Proposition 1.2.5] for Bochner integrals, since $\|f_N(x) - f(x)\|_{\mathfrak{H}} \leq \|f(x)\|_{\mathfrak{H}}$ a.e.

For fixed $0 < \beta < 1$ and $\gamma > 0$, we say that $\mathbf{f}$ defined on $\mathbb{R}$ belongs to $\mathcal{M}(\beta, \gamma, r, x_0; \mathfrak{H})$ with $r > 0$ and $x_0 \in \mathbb{R}$ if it satisfies $\int_{\mathbb{R}} \mathbf{f}(x) dx = 0$ and

$$(5.2) \quad \begin{aligned} |\mathbf{f}(x)|_{\mathfrak{H}} &\leq C \frac{r^\gamma}{(r + |x - x_0|)^{1+\gamma}}, \\ |\mathbf{f}(x) - \mathbf{f}(x')|_{\mathfrak{H}} &\leq C \left(\frac{|x - x'|}{r + |x - x_0|} \right)^\beta \frac{r^\gamma}{(r + |x - x_0|)^{1+\gamma}}, \quad \text{for } |x - x'| \leq \frac{r + |x - x_0|}{2}. \end{aligned}$$

Denote $\|\mathbf{f}\|_{\mathcal{M}(\beta, \gamma, r, x_0; \mathfrak{H})} := \inf\{C : (5.2) \text{ holds}\}$. It is a Banach space as in the scalar case. In particular, $\mathcal{M}(\beta, \gamma, r, x_0) = \mathcal{M}(\beta, \gamma, r, x_0; \mathbb{R})$.

Recall the Calderón reproducing formula (cf. e.g. [21, Section 2.3])

$$(5.3) \quad g = \int_0^{+\infty} \varphi_t * \rho_t * g \frac{dt}{t},$$

for $g \in L^2(\mathbb{R})$, where $\rho_t := -t \frac{\partial P_t}{\partial t}$, and $\varphi \in \mathcal{S}(\mathbb{R})$ is an even function satisfying

- (1) $\text{supp } \varphi \subseteq [-1, 1]$;
- (2) $\int_{\mathbb{R}} s^a \varphi(s) ds = 0$ for $a = 0, 1, 2$;
- (3) $2\pi \int_0^{+\infty} e^{-2\pi\xi} \widehat{\varphi}(\xi) d\xi = 1$.

Now define convolution $(\phi * \mathbf{f})(x) = \sum_i (\phi * f_i)(x) h_i$, for $\phi \in L^1(\mathbb{R})$ decaying like the Poisson kernel. Notice that $\phi * \mathbf{f} \in L^p(\mathbb{R}, \mathfrak{H})$ for $1 < p < \infty$ if $\mathbf{f} \in L^p(\mathbb{R}, \mathfrak{H})$. This is because

$$\begin{aligned} \|\phi * \mathbf{f}\|_{L^p(\mathbb{R}, \mathfrak{H})}^p &= \int_{\mathbb{R}} \left(\sum_i |\phi * f_i|^2(x) \right)^{\frac{p}{2}} dx \lesssim \int_{\mathbb{R}} \left(\sum_i |M(f_i)|^2(x) \right)^{\frac{p}{2}} dx \\ &\lesssim \int_{\mathbb{R}} \left(\sum_i |f_i|^2(x) \right)^{\frac{p}{2}} dx = \|\mathbf{f}\|_{L^p(\mathbb{R}, \mathfrak{H})}^p, \end{aligned}$$

by the vector-valued inequality for the Hardy-Littlewood maximal functions. We have the Calderón reproducing formula for $\mathfrak{H}$ -valued functions on $\mathbb{R}$:

Proposition 5.1. *Let $\varphi_t = \frac{1}{t} \varphi(\frac{\cdot}{t})$. For $\mathbf{f} \in L^2(\mathbb{R}, \mathfrak{H})$, we have*

$$(5.4) \quad \mathbf{f} = \lim_{\substack{T \rightarrow +\infty \\ \varepsilon \rightarrow +0}} \int_{\varepsilon}^T \varphi_t * \rho_t * \mathbf{f} \frac{dt}{t} =: \int_0^{+\infty} \varphi_t * \rho_t * \mathbf{f} \frac{dt}{t},$$

where the limit exists in the sense of $L^2(\mathbb{R}, \mathfrak{H})$.

Proof. For $0 < \varepsilon < T < \infty$, define first an operator on $L^2(\mathbb{R})$:

$$(5.5) \quad C_{\varepsilon, T}(g) := \int_{\varepsilon}^T \varphi_t * \rho_t * g \frac{dt}{t}, \quad g \in L^2(\mathbb{R}).$$

By Plancherel's theorem,

$$\|C_{\varepsilon, T}(g)\|_{L^2(\mathbb{R})} = \|m_{\varepsilon, T}(\xi) \widehat{g}(\xi)\|_{L^2(\mathbb{R})} \lesssim \|g\|_{L^2(\mathbb{R})},$$

where the implicit constant is independent of ε and T , and

$$m_{\varepsilon, T}(\xi) := \int_{\varepsilon}^T \widehat{\varphi}(t\xi) \widehat{\rho}_1(t\xi) \frac{dt}{t} = \int_{\varepsilon|\xi|}^{T|\xi|} \widehat{\varphi}(t) \widehat{\rho}_1(t) \frac{dt}{t}$$

is uniformly bounded. Now define operator on $L^2(\mathbb{R}, \mathfrak{H})$ by acting on each component:

$$C_{\varepsilon, T}(\mathfrak{f}) := \lim_{N \rightarrow +\infty} \sum_{i=1}^N C_{\varepsilon, T}(f_i) h_i = \lim_{N \rightarrow +\infty} \int_{\varepsilon}^T \varphi_t * \rho_t * \mathfrak{f}_N \frac{dt}{t}.$$

This series converges in $L^2(\mathbb{R}, \mathfrak{H})$, and $C_{\varepsilon, T}$ is uniformly bounded on $L^2(\mathbb{R}, \mathfrak{H})$ by

$$\|C_{\varepsilon, T}(\mathfrak{f})\|_{L^2(\mathbb{R}, \mathfrak{H})}^2 = \sum_{i=1}^{\infty} \|C_{\varepsilon, T}(f_i)\|_{L^2(\mathbb{R})}^2 \lesssim \sum_{i=1}^{\infty} \|f_i\|_{L^2(\mathbb{R})}^2 = \|\mathfrak{f}\|_{L^2(\mathbb{R}, \mathfrak{H})}^2.$$

It remains to prove the convergence to $\mathfrak{f}$. For each fixed N , the scalar Calderón formula applied to $f_1, \dots, f_N$ gives $C_{\varepsilon, T}(\mathfrak{f}_N) \rightarrow \mathfrak{f}_N$ as $T \rightarrow +\infty$ and $\varepsilon \rightarrow 0^+$ in $L^2(\mathbb{R}, \mathfrak{H})$. Using the uniform boundedness of $C_{\varepsilon, T}$, we obtain

$$\begin{aligned} \|C_{\varepsilon, T}(\mathfrak{f}) - \mathfrak{f}\|_{L^2(\mathbb{R}, \mathfrak{H})} &\leq \|C_{\varepsilon, T}(\mathfrak{f} - \mathfrak{f}_N)\|_{L^2(\mathbb{R}, \mathfrak{H})} + \|C_{\varepsilon, T}(\mathfrak{f}_N) - \mathfrak{f}_N\|_{L^2(\mathbb{R}, \mathfrak{H})} + \|\mathfrak{f}_N - \mathfrak{f}\|_{L^2(\mathbb{R}, \mathfrak{H})} \\ &\lesssim \|\mathfrak{f} - \mathfrak{f}_N\|_{L^2(\mathbb{R}, \mathfrak{H})} + \|C_{\varepsilon, T}(\mathfrak{f}_N) - \mathfrak{f}_N\|_{L^2(\mathbb{R}, \mathfrak{H})}. \end{aligned}$$

First choose N large enough so that $\|\mathfrak{f}_N - \mathfrak{f}\|_{L^2(\mathbb{R}, \mathfrak{H})}$ is small, and then let $T \rightarrow +\infty$ and $\varepsilon \rightarrow 0^+$. It follows that $C_{\varepsilon, T}(\mathfrak{f}) \rightarrow \mathfrak{f}$ in $L^2(\mathbb{R}, \mathfrak{H})$. $\square$

We have the following wavelet Calderón reproducing formula for $L^2(\mathbb{R}, \mathfrak{H})$ -functions. See the appendix for its proof. In the sequel, for $j \in \mathbb{Z}$, a *dyadic interval I with length $\ell(I) = 2^{-\alpha(j+N)}$* means $I = [0, 2^{-\alpha(j+N)}] + l2^{-\alpha(j+N)}$ for some $l \in \mathbb{Z}$. Here we take closed dyadic intervals for later convenience.

Proposition 5.2. *For fixed $0 < \beta, \gamma < 1$, there exist a fixed small $\alpha > 0$ and a large integer N such that for each dyadic interval I with length $\ell(I) = 2^{-\alpha(j+N)}$, $j \in \mathbb{Z}$, and every fixed point x_I in I , there exists $\phi_j(x, x_I) \in \mathcal{M}(\beta, \gamma, 2^{-\alpha j}, x_I; \mathfrak{H})$ with $\|\phi_j(x, x_I)\|_{\mathcal{M}(\beta, \gamma, 2^{-\alpha j}, x_I; \mathfrak{H})}$ uniformly bounded such that*

$$(5.6) \quad \mathfrak{f} = \sum_{j \in \mathbb{Z}} \sum_I \mathfrak{f}_{j; I} \phi_j(\cdot, x_I)$$

holds for $\mathfrak{f} \in L^2(\mathbb{R}, \mathfrak{H})$, where the summation is taken over dyadic intervals I with length $\ell(I) = 2^{-\alpha(j+N)}$, and

$$(5.7) \quad \mathfrak{f}_{j; I} = c_{\alpha} \ell(I) |\tilde{\rho}_j * \mathfrak{f}(x_I)|_{\mathfrak{H}}, \quad \tilde{\rho}_j := \frac{1}{c_{\alpha}} \int_{2^{-\alpha j}}^{2^{-\alpha(j-1)}} \rho_t \frac{dt}{t}, \quad c_{\alpha} = \int_{2^{-\alpha j}}^{2^{-\alpha(j-1)}} \frac{dt}{t} = \alpha \ln 2.$$

The series in (5.6) converges in $L^2(\mathbb{R}, \mathfrak{H})$.

We also need the Calderón reproducing formula with compactly supported ψ :

$$(5.8) \quad \mathfrak{f} = \int_0^{+\infty} \psi_t * \psi_t * \mathfrak{f} \frac{dt}{t},$$

for any $\mathfrak{f} \in L^2(\mathbb{R}, \mathfrak{H})$. Here we can require ψ to be smooth and compactly supported. This can be proved exactly in the same way as Proposition 5.1, just using the scalar Calderón reproducing formula with compactly supported ψ instead of (5.5). For $\mathfrak{H}$ -valued functions on $\mathbb{R}$, the S -function associated with ψ and g -function associated to the Poisson kernel are defined as

$$S_{\psi}(\mathfrak{f})(x) := \left(\int_0^{+\infty} \int_{|x-x'| < t} \|\psi_t * \mathfrak{f}\|_{\mathfrak{H}}^2(x') \frac{dx' dt}{t^2} \right)^{\frac{1}{2}} \quad \text{and} \quad g_P(\mathfrak{f})(x) := \left(\int_0^{+\infty} \|\rho_t * \mathfrak{f}\|_{\mathfrak{H}}^2(x) \frac{dt}{t} \right)^{\frac{1}{2}},$$

respectively, where $\psi_t(x) = \frac{1}{t}\psi(x/t)$. See, e.g. [23], for vector valued Littlewood-Paley g function and S functions. The wavelet Calderón reproducing formula (5.6) yields the following Plancherel-Pólya type inequality (cf. e.g. [21, Theorem 2.16-2.17] for the scalar case).

Proposition 5.3. *Let $\alpha > 0$ and N as in Proposition 5.2. For a fixed $C_0 > 0$ and every $f \in L^2(\mathbb{R}, \mathfrak{H})$,*

$$\left\| \left(\sum_{j \in \mathbb{Z}} \int_{2^{-\alpha j}}^{2^{-\alpha(j-1)}} \sup_{u \in C_0 I} \|\psi_t * f(u)\|_{\mathfrak{H}}^2 \frac{dt}{t} \chi_I(\cdot) \right)^{\frac{1}{2}} \right\|_{L^1} \lesssim \left\| \left(\sum_{j \in \mathbb{Z}} \inf_{u \in I} \int_{2^{-\alpha j}}^{2^{-\alpha(j-1)}} \|\rho_t * f(u)\|_{\mathfrak{H}}^2 \frac{dt}{t} \chi_I(\cdot) \right)^{\frac{1}{2}} \right\|_{L^1},$$

where the summation is taken over all dyadic intervals I with lengths $\ell(I) = 2^{-\alpha(j+N)}$, χ_I is the indicator function of I , and $C_0 I$ is the interval with the same mid-point as I and with length $C_0 \ell(I)$. The implicit constant depends on α, N and C_0 .

This inequality actually holds for certain distributions, but here the inequality for L^2 functions is sufficient for our purposes. See the Appendix A for a proof.

5.2. The estimate $\|f\|_{L^1} \lesssim \|g(f)\|_{L^1}$. We first use the Plancherel-Pólya type inequality to prove the control of the S -function by the g -function for $\mathfrak{H}$ -valued functions.

Proposition 5.4. *For $f \in L^2(\mathbb{R}, \mathfrak{H})$ with $g_P(f) \in L^1(\mathbb{R})$, we have $S_\psi(f) \in L^1(\mathbb{R})$ and $\|S_\psi(f)\|_{L^1(\mathbb{R})} \lesssim \|g_P(f)\|_{L^1(\mathbb{R})}$.*

Proof. We can write

$$S_\psi(f)(x) = \left(\sum_{j \in \mathbb{Z}} \sum_I \int_{2^{-\alpha j}}^{2^{-\alpha(j-1)}} \int_{\mathbb{R}} \chi_t(x - x') \|\psi_t * f(x')\|_{\mathfrak{H}}^2 \chi_I(x) \frac{dx' dt}{t} \right)^{\frac{1}{2}},$$

where the summation is taken over all dyadic intervals I with lengths $\ell(I) = 2^{-\alpha(j+N)}$. Note that there exists a fixed constant C_0 depending only on α, N such that for $2^{-\alpha j} \leq t \leq 2^{-\alpha(j-1)}$ and $x' \in \mathbb{R}$,

$$\chi_t(x - x') \|\psi_t * f(x')\|_{\mathfrak{H}}^2 \chi_I(x) \leq \chi_t(x - x') \sup_{u \in C_0 I} \|\psi_t * f(u)\|_{\mathfrak{H}}^2 \chi_I(x).$$

Therefore, we have

$$\begin{aligned} \|S_\psi(f)\|_{L^1(\mathbb{R})} &\leq \left\| \left(\sum_{j \in \mathbb{Z}} \sum_I \int_{2^{-\alpha j}}^{2^{-\alpha(j-1)}} \int_{\mathbb{R}} \chi_t(\cdot - x') \sup_{u \in C_0 I} \|\psi_t * f(u)\|_{\mathfrak{H}}^2 \chi_I(\cdot) \frac{dx' dt}{t} \right)^{\frac{1}{2}} \right\|_{L^1(\mathbb{R})} \\ &= \sqrt{2} \left\| \left(\sum_{j \in \mathbb{Z}} \sum_I \int_{2^{-\alpha j}}^{2^{-\alpha(j-1)}} \sup_{u \in C_0 I} \|\psi_t * f(u)\|_{\mathfrak{H}}^2 \chi_I(\cdot) \frac{dt}{t} \right)^{\frac{1}{2}} \right\|_{L^1(\mathbb{R})} \\ &\lesssim \left\| \left(\sum_{j \in \mathbb{Z}} \sum_I \inf_{u \in I} \int_{2^{-\alpha j}}^{2^{-\alpha(j-1)}} \|\rho_t * f(u)\|_{\mathfrak{H}}^2 \frac{dt}{t} \chi_I(\cdot) \right)^{\frac{1}{2}} \right\|_{L^1(\mathbb{R})} \\ &\leq \left\| \left(\sum_{j \in \mathbb{Z}} \sum_I \int_{2^{-\alpha j}}^{2^{-\alpha(j-1)}} \|\rho_t * f(\cdot)\|_{\mathfrak{H}}^2 \frac{dt}{t} \chi_I(\cdot) \right)^{\frac{1}{2}} \right\|_{L^1(\mathbb{R})} = \|g_P(f)\|_{L^1(\mathbb{R})}, \end{aligned}$$

by the Plancherel-Pólya type inequality in Proposition 5.3. $\square$

Proposition 5.5. *For any $\mathfrak{f} \in L^2(\mathbb{R}, \mathfrak{H})$ with $S_\psi(\mathfrak{f}) \in L^1(\mathbb{R})$, we have $\mathfrak{f} \in L^1(\mathbb{R}, \mathfrak{H})$ and $\|\mathfrak{f}\|_{L^1(\mathbb{R}, \mathfrak{H})} \lesssim \|S_\psi(\mathfrak{f})\|_{L^1(\mathbb{R})}$.*

Proof. This can be established by atomic decomposition for $\mathfrak{H}$ -valued L^2 functions with L^1 integrable S -functions, which follows from atomic decomposition of the vector-valued tent space T^1 (cf. [27, Theorem 4.5]) and the Calderón reproducing formula (5.8). We omit details. $\square$

Proposition 5.4 and 5.5 imply the following corollary.

Corollary 5.1. *For $\mathfrak{f} \in L^2(\mathbb{R}, \mathfrak{H})$ with $g_P(\mathfrak{f}) \in L^1(\mathbb{R})$, we have $\mathfrak{f} \in L^1(\mathbb{R}, \mathfrak{H})$ and $\|\mathfrak{f}\|_{L^1(\mathbb{R}, \mathfrak{H})} \lesssim \|g_P(\mathfrak{f})\|_{L^1(\mathbb{R})}$.*

Proposition 5.6. *For $f \in L^2(\mathbb{R}^n)$ with $g(f) \in L^1(\mathbb{R}^n)$, we have $f \in L^1(\mathbb{R}^n)$ and $\|f\|_{L^1(\mathbb{R}^n)} \lesssim \|g(f)\|_{L^1(\mathbb{R}^n)}$.*

Proof. Let $\mathfrak{H} := L^2((\mathbb{R}_+)^{m-1}, \frac{d\mathbf{t}_{\mathbf{m}_2}}{\mathbf{t}_{\mathbf{m}_2}})$ with the usual L^2 -norm. It is a separable Hilbert space. By Proposition 3.3 and the estimate (3.15), $f_\varepsilon := f * P_\varepsilon$ is smooth and belongs to $L^2 \cap L^\infty(\mathbb{R}^n)$. Moreover, $g(f_\varepsilon) \in L^1(\mathbb{R}^n)$ since

$$(5.9) \quad g(f_\varepsilon) = \left(\int_{(\mathbb{R}_+)^m} |\nabla_1 \cdots \nabla_m (f * P_{\mathbf{t}+\varepsilon})|^2(x) \mathbf{t} d\mathbf{t} \right)^{1/2} \leq g(f)$$

by definition (1.10) and changing variables. So we can assume at first that f is smooth and belongs to $L^2 \cap L^\infty(\mathbb{R}^n)$.

Note that f is measurable in each line by continuity, and $f *_{\mu} t_\mu \partial_{t_\mu} P_{t_\mu}(x)$ exists at every (x, t_μ) by the boundedness of f . It is direct to check $f *_{\mu} t_\mu \partial_{t_\mu} P_{t_\mu}(x)$ is continuous in (x, t_μ) and belongs to $L^2 \cap L^\infty(\mathbb{R}^n)$ uniformly. Repeating this procedure, we see that $f *_{\mu} t_\mu \partial_{t_\mu} P_{t_\mu} * \cdots *_{\mathbf{m}} t_{\mathbf{m}} \partial_{t_{\mathbf{m}}} P_{t_{\mathbf{m}}}(x)$ is continuous in $(x, t_\mu, \dots, t_{\mathbf{m}})$. Now define measurable functions

$$\mathfrak{f}_{x^\perp}(s)(t_2, \dots, t_{\mathbf{m}}) := f *_{\mathbf{2}} t_2 \partial_{t_2} P_{t_2} * \cdots *_{\mathbf{m}} t_{\mathbf{m}} \partial_{t_{\mathbf{m}}} P_{t_{\mathbf{m}}}(x^\perp + se_1).$$

For almost all $x^\perp \in e_1^\perp$, $\mathfrak{f}_{x^\perp}$ is $\mathfrak{H}$ -valued because

$$(5.10) \quad \begin{aligned} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \|\mathfrak{f}_{x^\perp}(s)\|_{\mathfrak{H}}^2 ds dx^\perp &= \int_{\mathbb{R}^n} \int_{(\mathbb{R}_+)^{m-1}} |f *_{\mathbf{2}} t_2 \partial_{t_2} P_{t_2} * \cdots *_{\mathbf{m}} t_{\mathbf{m}} \partial_{t_{\mathbf{m}}} P_{t_{\mathbf{m}}}(x^\perp + se_1)|^2 \frac{d\mathbf{t}_{\mathbf{m}_2}}{\mathbf{t}_{\mathbf{m}_2}} ds dx^\perp \\ &\leq \|g_{\mathbf{m}_2}(f)\|_{L^2}^2 \lesssim \|f\|_{L^2}^2, \end{aligned}$$

by Proposition 2.2. Consequently, $\mathfrak{f}_{x^\perp} \in L^2(\mathbb{R}, \mathfrak{H})$ for almost all $x^\perp \in e_1^\perp$. On the other hand,

$$(5.11) \quad \begin{aligned} g_P(\mathfrak{f}_{x^\perp})(s) &= \left(\int_{\mathbb{R}_+} \|\mathfrak{f}_{x^\perp} * t_1 \partial_{t_1} P_{t_1}(s)\|_{\mathfrak{H}}^2 \frac{dt_1}{t_1} \right)^{\frac{1}{2}} \\ &= \left(\int_{(\mathbb{R}_+)^m} |f *_{\mathbf{2}} t_2 \partial_{t_2} P_{t_2} * \cdots *_{\mathbf{m}} t_{\mathbf{m}} \partial_{t_{\mathbf{m}}} P_{t_{\mathbf{m}}} *_{\mathbf{1}} t_1 \partial_{t_1} P_{t_1}(x^\perp + se_1)|^2 \frac{d\mathbf{t}_{\mathbf{m}_2}}{\mathbf{t}_{\mathbf{m}_2}} \frac{dt_1}{t_1} \right)^{\frac{1}{2}} \\ &= \left(\int_{(\mathbb{R}_+)^m} |f *_{\mathbf{1}} \partial_{t_1} P_{t_1} *_{\mathbf{2}} \cdots *_{\mathbf{m}} \partial_{t_{\mathbf{m}}} P_{t_{\mathbf{m}}}|^2(x^\perp + se_1) \mathbf{t} d\mathbf{t} \right)^{\frac{1}{2}} \leq g(f)(x^\perp + se_1), \end{aligned}$$

by the commutativity (2.4) of partial convolutions along lines. Consequently, by $g(f) \in L^1(\mathbb{R}^n)$, we see that $g_P(\mathbf{f}_{x^\perp}) \in L^1(\mathbb{R})$ for almost all $x^\perp \in e_1^\perp$. Now applying Corollary 5.1, we get $\mathbf{f}_{x^\perp} \in L^1(\mathbb{R}, \mathfrak{H})$ for almost all $x^\perp \in e_1^\perp$, and

$$\int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \|\mathbf{f}_{x^\perp}\|_{\mathfrak{H}}(s) ds dx^\perp \lesssim \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} g_P(\mathbf{f}_{x^\perp})(s) ds dx^\perp \leq \int_{\mathbb{R}^n} g(f)(x^\perp + se_1) ds dx^\perp.$$

It can be rewritten as

$$\int_{\mathbb{R}^n} \left(\int_{(\mathbb{R}_+)^{m-1}} |f *_{\mathbf{t}_2} t_2 \partial_{t_2} P_{t_2} * \cdots *_{\mathbf{t}_m} t_m \partial_{t_m} P_{t_m}(x)|^2 \frac{d\mathbf{t}_{\mathbf{m}_2}}{\mathbf{t}_{\mathbf{m}_2}} \right)^{\frac{1}{2}} dx \lesssim \|g(f)\|_{L^1(\mathbb{R}^n)}.$$

If we define $\mathfrak{H}_1 := L^2((\mathbb{R}_+)^{m-2}, \frac{d\mathbf{t}_{\mathbf{m}_3}}{\mathbf{t}_{\mathbf{m}_3}})$ with the usual norm, then as above,

$$\mathbf{f}_{1;x'}(s) = f *_{\mathbf{t}_3} t_3 \partial_{t_3} P_{t_3} * \cdots *_{\mathbf{t}_m} t_m \partial_{t_m} P_{t_m}(x' + se_2)$$

is measurable and defines functions $\mathbf{f}_{1;x'} : \mathbb{R} \rightarrow \mathfrak{H}_1$ for almost all $x' \in e_2^\perp$. Similarly, as in (5.10)-(5.11), we get $\mathbf{f}_{1;x'} \in L^2(\mathbb{R}, \mathfrak{H}_1)$ and $g_P(\mathbf{f}_{1;x'}) \in L^1(\mathbb{R})$ for almost all $x' \in e_2^\perp$, and

$$\begin{aligned} & \int_{\mathbb{R}^n} \left(\int_{(\mathbb{R}_+)^{m-2}} |f *_{\mathbf{t}_3} t_3 \partial_{t_3} P_{t_3} * \cdots *_{\mathbf{t}_m} t_m \partial_{t_m} P_{t_m}(x)|^2 \frac{d\mathbf{t}_{\mathbf{m}_3}}{\mathbf{t}_{\mathbf{m}_3}} \right)^{\frac{1}{2}} dx \\ & \lesssim \int_{\mathbb{R}^n} \left(\int_{(\mathbb{R}_+)^{m-1}} |f *_{\mathbf{t}_2} t_2 \partial_{t_2} P_{t_2} * \cdots *_{\mathbf{t}_m} t_m \partial_{t_m} P_{t_m}(x)|^2 \frac{d\mathbf{t}_{\mathbf{m}_2}}{\mathbf{t}_{\mathbf{m}_2}} \right)^{\frac{1}{2}} dx \lesssim \|g(f)\|_{L^1(\mathbb{R}^n)}. \end{aligned}$$

Repeating this procedure, we finally get

$$\|f\|_{L^1(\mathbb{R}^n)} \lesssim \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}_+} |f *_{\mathbf{t}_m} t_m \partial_{t_m} P_{t_m}(x)|^2 \frac{dt_m}{t_m} \right)^{\frac{1}{2}} dx \lesssim \|g(f)\|_{L^1(\mathbb{R}^n)}.$$

For the original function f , apply the preceding estimate to $f_\varepsilon = f * P_\varepsilon$ to get

$$\|f_\varepsilon\|_{L^1(\mathbb{R}^n)} \lesssim \|g(f_\varepsilon)\|_{L^1(\mathbb{R}^n)} \lesssim \|g(f)\|_{L^1(\mathbb{R}^n)},$$

by (5.9). Since $f \in L^2(\mathbb{R}^n)$, we have $f_\varepsilon \rightarrow f$ in $L^2(\mathbb{R}^n)$ by Proposition 3.3 (a). Hence, after passing to a subsequence if necessary, $f_\varepsilon(x) \rightarrow f(x)$ for a.e. x . By Fatou's lemma,

$$\|f\|_{L^1(\mathbb{R}^n)} \leq \liminf_{\varepsilon \rightarrow 0} \|f_\varepsilon\|_{L^1(\mathbb{R}^n)} \lesssim \|g(f)\|_{L^1(\mathbb{R}^n)}.$$

The proposition is proved. $\square$

Corollary 5.2. *For $f \in L^2(\mathbb{R}^n)$ with $g(f) \in L^1(\mathbb{R}^n)$, we have $f \in L^1(\mathbb{R}^n)$ and*

$$\sup_{\mathbf{t} \in (\mathbb{R}_+)^m} \|f * P_{\mathbf{t}}\|_{L^1(\mathbb{R}^n)} \lesssim \|g(f)\|_{L^1(\mathbb{R}^n)}.$$

Proof. Since $\|\pi(P_{t_1} \cdots P_{t_m})\|_{L^1(\mathbb{R}^n)} = 1$ for every $\mathbf{t} \in (\mathbb{R}_+)^m$ by (3.13), we have $\|f * P_{\mathbf{t}}\|_{L^1(\mathbb{R}^n)} \leq \|f\|_{L^1(\mathbb{R}^n)}$ by Young's inequality. Then the result follows from Proposition 5.6. $\square$

6. PROOF OF THE EQUIVALENCE OF VARIOUS CHARACTERIZATIONS

In this section, we prove the equivalence of various characterizations in Theorem 1.2.

(1) implies (2). Let $f_\varepsilon := F(\cdot + \mathbf{i}\pi(\varepsilon)) \in L^1(\mathbb{R}^n)$. Then, $|f_\varepsilon|^q \in L^r(\mathbb{R}^n)$ with $r = \frac{1}{q} > 1$, if we choose $0 < q < 1$. Apply Proposition 3.6 to get

$$|F(x + \mathbf{i}\pi(\mathbf{t} + \varepsilon))|^q \leq |f_\varepsilon|^q * P_{\mathbf{t}}(x)$$

for $(x, \mathbf{t}) \in \mathbb{R}^n \times (\mathbb{R}_+)^m$. Then

$$\sup_{(x', \mathbf{t}) \in \Gamma_\beta(x)} |F(x' + \mathbf{i}\pi(\mathbf{t} + \varepsilon))|^q \leq \sup_{(x', \mathbf{t}) \in \Gamma_\beta(x)} |f_\varepsilon|^q * P_{\mathbf{t}}(x') \lesssim M_{\text{it}}(|f_\varepsilon|^q)(x),$$

by the analogue of (3.1) for $\Gamma_\beta(x)$, where M_{it} is the iterated maximal function on $\mathbb{R}^n$. Therefore,

$$(6.1) \quad \left\| \sup_{(x', \mathbf{t}) \in \Gamma_\beta(x)} |F(x' + \mathbf{i}\pi(\mathbf{t} + \varepsilon))|^q \right\|_{L^r(\mathbb{R}^n, dx)}^r \lesssim \|M_{\text{it}}(|f_\varepsilon|^q)\|_{L^r(\mathbb{R}^n)}^r \lesssim \| |f_\varepsilon|^q \|_{L^r(\mathbb{R}^n)}^r = \|f_\varepsilon\|_{L^1(\mathbb{R}^n)}^r,$$

where implicit constants are independent of F and ε . Letting $\varepsilon \rightarrow 0$ in (6.1), we obtain

$$(6.2) \quad \begin{aligned} \left\| \sup_{(x', \mathbf{t}) \in \Gamma_\beta(x)} |F(x' + \mathbf{i}\pi(\mathbf{t}))| \right\|_{L^1(\mathbb{R}^n, dx)} &= \left\| \lim_{\varepsilon \rightarrow 0} \sup_{(x', \mathbf{t}) \in (0, \varepsilon) + \Gamma_\beta(x)} |F(x' + \mathbf{i}\pi(\mathbf{t}))|^q \right\|_{L^r(\mathbb{R}^n, dx)}^r \\ &\leq \liminf_{\varepsilon \rightarrow 0} \left\| \sup_{(x', \mathbf{t}) \in \Gamma_\beta(x)} |F(x' + \mathbf{i}\pi(\mathbf{t} + \varepsilon))|^q \right\|_{L^r(\mathbb{R}^n, dx)}^r \\ &\lesssim \liminf_{\varepsilon \rightarrow 0} \|f_\varepsilon\|_{L^1(\mathbb{R}^n)} \end{aligned}$$

by Fatou's lemma, since domains $(0, \varepsilon) + \Gamma_\beta(x)$ are increasing when $\varepsilon \downarrow 0$ and the limit is the set $\Gamma_\beta(x)$. Since $\pi(\varepsilon) \in \Omega$, we have $\|f_\varepsilon\|_{L^1(\mathbb{R}^n)} = \|F(\cdot + \mathbf{i}\pi(\varepsilon))\|_{L^1(\mathbb{R}^n)} \leq \|F\|_{H^1(T_\Omega)}$. Therefore, $\|\mathbb{N}^\beta(F)\|_{L^1(\mathbb{R}^n)} \lesssim \|F\|_{H^1(T_\Omega)}$.

(2) implies (3). First, $|F(x + \mathbf{i}y)| \leq \mathbb{N}^\beta(F)(x)$ by the definition of non-tangential maximal function. Hence,

$$\sup_{y \in \Omega} \int_{\mathbb{R}^n} |F(x + \mathbf{i}y)| dx \leq \int_{\mathbb{R}^n} \mathbb{N}^\beta(F)(x) dx.$$

Thus, a holomorphic function F on T_Ω satisfying $\mathbb{N}^\beta(F) \in L^1(\mathbb{R}^n)$ belongs to $H^1(T_\Omega)$. Consequently, $F(x + \mathbf{i}y) \rightarrow 0$ as $|y| \rightarrow +\infty$ in $y_0 + \Omega$ for any fixed $y_0 \in \Omega$, by the boundary growth for H^1 holomorphic functions in Proposition 3.5. Now we can apply Theorem 1.1 to get $F(x + \mathbf{i}\pi(\mathbf{t})) = F^b * P_{\mathbf{t}}$ with $F^b \in L^1(\mathbb{R}^n)$. Then, it follows from Corollary 4.1 that $\|\mathbb{S}(F)\|_{L^1(\mathbb{R}^n)} = \|S(F^b)\|_{L^1(\mathbb{R}^n)} \lesssim \|N^\beta(F^b)\|_{L^1(\mathbb{R}^n)} = \|\mathbb{N}^\beta(F)\|_{L^1(\mathbb{R}^n)}$.

(3) implies (4).

We first prove a real version of pointwise control of the g -function by the S -function.

Proposition 6.1. *For $f \in L^1(\mathbb{R}^n)$, we have $g(f)(x) \lesssim S(f)(x)$.*

Proof. By translation invariance, it is sufficient to prove the inequality at $x = 0$. Let $u(x, \mathbf{t}) := f * P_{\mathbf{t}}(x)$, $x \in \mathbb{R}^n$, $\mathbf{t} \in (\mathbb{R}_+)^m$. By Proposition 2.1, we have $\Delta_\mu u(x, \mathbf{t}) = 0$, $\mu = 1, \dots, m$. Thus, each

component of $\nabla_{\mathbf{m}} u(x, \mathbf{t})$ is harmonic in the half-plane spanned by e_μ and t_μ . Hence, for fixed $\mathbf{t} \in (\mathbb{R}_+)^m$, repeatedly using the mean value formula gives

$$(6.3) \quad \nabla_{\mathbf{m}} u(0, \mathbf{t}) = \frac{2^{2m}}{\pi^m \gamma_0^{2m} t_1^2 \cdots t_m^2} \int_{D_1(\gamma_0 t_1/2) \times \cdots \times D_m(\gamma_0 t_m/2)} \nabla_{\mathbf{m}} u \left(\sum_{j=1}^m s'_j e_j, \mathbf{t}' \right) ds' d\mathbf{t}',$$

where $D_j(r) := \{(s'_j, t'_j) : |s'_j|^2 + |t'_j - t_j|^2 \leq r^2\}$. Notice that, if $(s'_j, t'_j) \in D_j(\gamma_0 t_j/2)$, then $t_j \approx t'_j$ and $|s'_j| \leq \gamma_0 t'_j$. Applying the Cauchy–Schwarz inequality to (6.3), and then using Fubini's theorem, we obtain

$$\begin{aligned} g^2(f)(0) &= \int_{(\mathbb{R}_+)^m} |\nabla_{\mathbf{m}} u(0, \mathbf{t})|^2 \mathbf{t} d\mathbf{t} \\ &\lesssim \int_{(\mathbb{R}_+)^m} \int_{D_1(\gamma_0 t_1/2) \times \cdots \times D_m(\gamma_0 t_m/2)} \left| \nabla_{\mathbf{m}} u \left(\sum_{j=1}^m s'_j e_j, \mathbf{t}' \right) \right|^2 ds' d\mathbf{t}' \frac{d\mathbf{t}}{t_1 \cdots t_m} \\ &\lesssim \int_{(\mathbb{R}_+)^m} \int_{|s'_1| \leq \gamma_0 t'_1} \cdots \int_{|s'_m| \leq \gamma_0 t'_m} \left| \nabla_{\mathbf{m}} u \left(\sum_{j=1}^m s'_j e_j, \mathbf{t}' \right) \right|^2 ds' d\mathbf{t}' \end{aligned}$$

Then, by Proposition 3.2, we get

$$\begin{aligned} g^2(f)(0) &\lesssim \int_{(\mathbb{R}_+)^m} |\nabla_{\mathbf{m}} u(\cdot, \mathbf{t}')|^2 * \chi_{\gamma_0 \mathbf{t}'}(0) \mathbf{t}' d\mathbf{t}' \\ &\lesssim \int_{(\mathbb{R}_+)^m} \int_{R(0, \mathbf{t}')} |\nabla_{\mathbf{m}} u(x', \mathbf{t}')|^2 dx' \frac{\mathbf{t}' d\mathbf{t}'}{|R(0, \mathbf{t}')|} \\ &= \int_{\Gamma(0)} |\nabla_{\mathbf{m}} u(x', \mathbf{t}')|^2 \frac{\mathbf{t}' d\mathbf{t}' dx'}{|R(0, \mathbf{t}')|} = S^2(f)(0). \end{aligned}$$

Here the last inequality follows from Proposition 3.2 and (3.5). Therefore $g(f)(0) \lesssim S(f)(0)$. By translation invariance, we get $g(f)(x) \lesssim S(f)(x)$, for $x \in \mathbb{R}^n$. The proof is complete. $\square$

The pointwise control of the $\mathbb{G}$ -function by the $\mathbb{S}$ -function for holomorphic functions follows by applying the same argument to $u(x, \mathbf{t}) = F(x + i\pi(\mathbf{t}))$, since u is harmonic in every half-plane spanned by e_μ and t_μ by Proposition 2.1 (2), and so is each component of $\nabla_{\mathbf{m}} u$. Hence $\mathbb{G}(F)(x) \lesssim \mathbb{S}(F)(x)$.

(4) implies (1). For $0 < \varepsilon < M$, let

$$(6.4) \quad \begin{aligned} F_{\varepsilon M}(x + i\pi(\mathbf{t})) &:= \sum_{\mathbf{j}} (-1)^{|\mathbf{j}|} F(x + i\pi(\mathbf{t} + \mathbf{M}_{\mathbf{j}})) \\ &= \int_{t_1+\varepsilon}^{t_1+M} \cdots \int_{t_m+\varepsilon}^{t_m+M} \partial_{t'_1} \cdots \partial_{t'_m} (F(x + i\pi(\mathbf{t}')))) dt', \end{aligned}$$

where the summation is taken over all subsets $\mathbf{j}$ of $\{1, 2, \dots, m\}$, and

$$\mathbf{M}_{\mathbf{j}} = (M_1, \dots, M_m), \quad M_k = \begin{cases} M, & k \notin \mathbf{j}, \\ \varepsilon, & k \in \mathbf{j}. \end{cases}$$

By applying the Cauchy–Schwarz inequality to (6.4), we get

$$\begin{aligned} |F_{\varepsilon M}(x + \mathbf{i}\pi(\mathbf{t}))| &\leq \left(\ln \frac{M}{\varepsilon}\right)^{\frac{m}{2}} \left(\int_{t_1+\varepsilon}^{t_1+M} \cdots \int_{t_m+\varepsilon}^{t_m+M} |\partial_{t'_1} \cdots \partial_{t'_m}(F(x + \mathbf{i}\pi(\mathbf{t}')))|^2 d\mathbf{t}'\right)^{\frac{1}{2}} \\ &\leq \left(\ln \frac{M}{\varepsilon}\right)^{\frac{m}{2}} \mathbb{G}(F)(x) < \infty. \end{aligned}$$

Thus,

$$(6.5) \quad \sup_{\mathbf{t} \in (\mathbb{R}_+)^m} \|F_{\varepsilon M}(\cdot + \mathbf{i}\pi(\mathbf{t}))\|_{L^1(\mathbb{R}^n)} \leq \left(\ln \frac{M}{\varepsilon}\right)^{\frac{m}{2}} \|\mathbb{G}(F)\|_{L^1(\mathbb{R}^n)} < +\infty.$$

From (6.4) we can see that

$$(6.6) \quad F_{\varepsilon M}(x + \mathbf{i}y) = \sum_{\mathbf{j}} (-1)^{|\mathbf{j}|} F(x + \mathbf{i}y + \mathbf{i}\pi(\mathbf{M}_{\mathbf{j}}))$$

is holomorphic on T_{Ω} . Thus, $F_{\varepsilon M} \in H^1(T_{\Omega})$ by (6.5). Then, by Theorem 1.1, we get

$$F_{\varepsilon M}(x + \mathbf{i}\pi(\mathbf{t})) = f_{\varepsilon M} * P_{\mathbf{t}}(x)$$

with some $f_{\varepsilon M} \in L^1(\mathbb{R}^n)$. Fix $\delta > 0$ and write $\boldsymbol{\delta} := (\delta, \dots, \delta) \in (\mathbb{R}_+)^m$. Define

$$(6.7) \quad f_{\varepsilon M, \delta} := f_{\varepsilon M} * P_{\boldsymbol{\delta}} = F_{\varepsilon M}(\cdot + \mathbf{i}\pi(\boldsymbol{\delta})).$$

Since $F_{\varepsilon M} \in H^1(T_{\Omega})$, by applying Proposition 3.5 to $F_{\varepsilon M}$ at the fixed height $\pi(\boldsymbol{\delta})$, we see that $f_{\varepsilon M, \delta} \in L^\infty(\mathbb{R}^n)$. Moreover, $f_{\varepsilon M, \delta} \in L^1(\mathbb{R}^n)$ by (6.7) and Young's inequality. Hence $f_{\varepsilon M, \delta} \in L^2(\mathbb{R}^n)$.

On the other hand, by the definition (6.4) of $F_{\varepsilon M}$ and a change of variables,

$$(6.8) \quad \mathbb{G}(F_{\varepsilon M})(x) \lesssim \sum_{\mathbf{j}} \left(\int_{(\mathbb{R}_+)^m} |\nabla_{\mathbf{m}}(F(x + \mathbf{i}\pi(\mathbf{t} + \mathbf{M}_{\mathbf{j}})))|^2 d\mathbf{t} \right)^{\frac{1}{2}} \lesssim \mathbb{G}(F)(x).$$

Furthermore, using (6.7) and the semigroup property of the Poisson kernels, we obtain

$$(6.9) \quad g(f_{\varepsilon M, \delta})(x) = \left(\int_{(\mathbb{R}_+)^m} |\nabla_{\mathbf{m}}(F_{\varepsilon M}(x + \mathbf{i}\pi(\mathbf{t} + \boldsymbol{\delta})))|^2 d\mathbf{t} \right)^{\frac{1}{2}} \leq \mathbb{G}(F_{\varepsilon M})(x) \lesssim \mathbb{G}(F)(x),$$

by setting $\mathbf{s} = \mathbf{t} + \boldsymbol{\delta}$ and using $t_1 \cdots t_m \leq s_1 \cdots s_m$. Thus, by applying Corollary 5.2 to $f_{\varepsilon M, \delta} \in L^2(\mathbb{R}^n)$, we get for every $\mathbf{t} \in (\mathbb{R}_+)^m$,

$$(6.10) \quad \|F_{\varepsilon M}(\cdot + \mathbf{i}\pi(\mathbf{t} + \boldsymbol{\delta}))\|_{L^1(\mathbb{R}^n)} = \|f_{\varepsilon M, \delta} * P_{\mathbf{t}}\|_{L^1(\mathbb{R}^n)} \lesssim \|g(f_{\varepsilon M, \delta})\|_{L^1(\mathbb{R}^n)} \lesssim \|\mathbb{G}(F)\|_{L^1(\mathbb{R}^n)}.$$

The implicit constants are independent of δ, ε, M , and $\mathbf{t}$. Letting $\delta \rightarrow 0^+$ in (6.10) and applying Fatou's lemma, we conclude that $\|F_{\varepsilon M}(\cdot + \mathbf{i}\pi(\mathbf{t}))\|_{L^1(\mathbb{R}^n)} \lesssim \|\mathbb{G}(F)\|_{L^1(\mathbb{R}^n)}$ for any $\mathbf{t} \in (\mathbb{R}_+)^m$.

Since for $y \in \Omega$, $F(x + \mathbf{i}y + \mathbf{i}y') \rightarrow 0$ as $|y'| \rightarrow +\infty$ by the assumption, it follows from (6.6) that $F_{\varepsilon M}(x + \mathbf{i}y) \rightarrow (-1)^m F(x + \mathbf{i}y)$ as $\varepsilon \rightarrow 0$ and $M \rightarrow +\infty$. So we conclude that

$$\|F(\cdot + \mathbf{i}y)\|_{L^1(\mathbb{R}^n)} \leq \liminf_{\substack{\varepsilon \rightarrow 0 \\ M \rightarrow +\infty}} \|F_{\varepsilon M}(\cdot + \mathbf{i}y)\|_{L^1(\mathbb{R}^n)} \lesssim \|\mathbb{G}(F)\|_{L^1(\mathbb{R}^n)},$$

by Fatou's lemma, where the implicit constants are independent of $y \in \Omega$. This completes the proof.

Proof of Corollary 1.1. By Theorem 1.1, if $F \in H^1(T_\Omega)$, then F has a boundary value $F^b \in L^1(\mathbb{R}^n)$ and $F(x + i\pi(\mathbf{t})) = F^b * P_{\mathbf{t}}(x)$, $x \in \mathbb{R}^n$, $\mathbf{t} \in (\mathbb{R}_+)^m$. Hence, by the definitions,

$$N^\beta(F^b) = \mathbb{N}^\beta(F), \quad S(F^b) = \mathbb{S}(F), \quad g(F^b) = \mathbb{G}(F).$$

Since $F \in H^1(T_\Omega)$, Theorem 1.2 implies that $\mathbb{N}^\beta(F)$, $\mathbb{S}(F)$ and $\mathbb{G}(F)$ belong to $L^1(\mathbb{R}^n)$, with fixed $\beta > 0$. Therefore, $N^\beta(F^b)$, $S(F^b)$ and $g(F^b)$ also belong to $L^1(\mathbb{R}^n)$. Together with $F^b \in L^1(\mathbb{R}^n)$, this proves the corollary. $\square$

APPENDIX A. THE PLANCHEREL-PÓLYA TYPE INEQUALITY FOR $\mathfrak{H}$ -VALUED FUNCTIONS

See e.g. [14, 19, 20, 21] for Plancherel-Pólya type inequality. Recall that for $\mathfrak{f}(x) = \sum_{j=1}^\infty f_j(x)h_j$, $\mathfrak{f}_k(x) = \sum_{j=1}^k f_j(x)h_j$ converges to $\mathfrak{f}(x)$ under the norm $L^p(\mathbb{R}, \mathfrak{H})$, i.e.

$$\int_{\mathbb{R}} \|\mathfrak{f}(x) - \mathfrak{f}_k(x)\|_{\mathfrak{H}}^p dx = \int_{\mathbb{R}} \left(\sum_{j=k+1}^\infty |f_j(x)|^2 \right)^{\frac{p}{2}} dx \rightarrow 0.$$

By the wavelet Calderón reproducing formula (5.4) on $\mathbb{R}$, we get

$$\begin{aligned} \mathfrak{f}(x) &= \int_{\mathbb{R}_+} \varphi_t * \rho_t * \mathfrak{f}(x) \frac{dt}{t} = \sum_{j \in \mathbb{Z}} \int_{2^{-\alpha j}}^{2^{-\alpha(j-1)}} \varphi_t * \rho_t * \mathfrak{f}(x) \frac{dt}{t} \\ (A.1) \quad &= c_\alpha \sum_j \tilde{\varphi}_j * \tilde{\rho}_j * \mathfrak{f}(x) + \sum_{j \in \mathbb{Z}} \int_{2^{-\alpha j}}^{2^{-\alpha(j-1)}} (\varphi_t * \rho_t - \tilde{\varphi}_j * \tilde{\rho}_j) * \mathfrak{f}(x) \frac{dt}{t} := \mathcal{T}_\alpha \mathfrak{f}(x) + \mathcal{R}_\alpha \mathfrak{f}(x), \end{aligned}$$

for $\mathfrak{f} \in L^2(\mathbb{R}, \mathfrak{H})$, where

$$(A.2) \quad \tilde{\varphi}_j := \frac{1}{c_\alpha} \int_{2^{-\alpha j}}^{2^{-\alpha(j-1)}} \varphi_t \frac{dt}{t}.$$

The second identity in (A.1) is subject to the condition (to be proved later) that both $\mathcal{T}_\alpha \mathfrak{f}(x)$ and $\mathcal{R}_\alpha \mathfrak{f}(x)$ are well defined. Now cover $\mathbb{R}$ by almost disjoint dyadic intervals I (only endpoints may overlap) with $\ell(I) = 2^{-\alpha(j+N)}$, where N is a large positive integer fixed later. Then

$$\begin{aligned} \mathcal{T}_\alpha \mathfrak{f}(x) &:= c_\alpha \sum_{j \in \mathbb{Z}} \tilde{\varphi}_j * \tilde{\rho}_j * \mathfrak{f}(x) = c_\alpha \sum_{j \in \mathbb{Z}} \sum_I \int_I \tilde{\varphi}_j(x - x') \tilde{\rho}_j * \mathfrak{f}(x') dx' \\ (A.3) \quad &= c_\alpha \sum_{j \in \mathbb{Z}} \sum_I \ell(I) \left(\frac{1}{\ell(I)} \int_I \tilde{\varphi}_j(x - x') dx' \right) \tilde{\rho}_j * \mathfrak{f}(x_I) + \mathcal{R}'_{\alpha, N} \mathfrak{f}(x) \end{aligned}$$

with x_I any fixed point in I , and

$$(A.4) \quad \mathcal{R}'_{\alpha, N} \mathfrak{f}(x) := c_\alpha \sum_{j \in \mathbb{Z}} \sum_I \int_I \tilde{\varphi}_j(x - x') \left(\tilde{\rho}_j * \mathfrak{f}(x') - \tilde{\rho}_j * \mathfrak{f}(x_I) \right) dx'.$$

Now we can write

$$(A.5) \quad \mathfrak{f}(x) = \mathcal{T} \mathfrak{f}(x) + \mathcal{R}_\alpha \mathfrak{f}(x) + \mathcal{R}'_{\alpha, N} \mathfrak{f}(x), \quad \text{where} \quad \mathcal{T} \mathfrak{f}(x) = \sum_j \sum_I \mathfrak{f}_{j;I} \tilde{\phi}_j(\cdot, x_I),$$

where $\mathfrak{f}_{j;I}$'s are given by (5.7), and

$$(A.6) \quad \tilde{\phi}_j(x, x_I) := \frac{1}{\ell(I)} \int_I \tilde{\varphi}_j(x - x') dx' \frac{\tilde{\rho}_j * \mathfrak{f}(x_I)}{|\tilde{\rho}_j * \mathfrak{f}(x_I)|_{\mathfrak{H}}} \in \mathcal{M}(\beta, \gamma, 2^{-\alpha j}, x_I; \mathfrak{H}),$$

(see [21] for the scalar case). Here we take $\tilde{\phi}_j(\cdot, x_I) \equiv 0$ if $|\tilde{\rho}_j * \mathbf{f}(x_I)|_{\mathfrak{H}} = 0$. Since φ is compactly supported and smooth, it is easy to see that $\tilde{\phi}_j(\cdot, x_I)$ have uniformly bounded norm in $\mathcal{M}(\beta, \gamma, 2^{-\alpha j}, x_I; \mathfrak{H})$ by the scalar case. In order for (A.3)–(A.5) to make sense and to invert $\mathcal{T}$, we need to show that $\mathcal{R}_\alpha$ and $\mathcal{R}'_{\alpha, N}$ have sufficiently small operator norms both on $L^2(\mathbb{R}, \mathfrak{H})$ and on $\mathcal{M}(\beta, \gamma, r, x_0; \mathfrak{H})$. Then, the series in (A.1) converges in $L^2(\mathbb{R}, \mathfrak{H})$. Here $\mathcal{R}'_{\alpha, N}$ has kernel

$$(A.7) \quad \mathcal{R}'_{\alpha, N}(x, u) = \sum_j \int_{\mathbb{R}} \tilde{\varphi}_j(x - x') \mathcal{R}_{\alpha, N; j}(x', u) dx',$$

with

$$(A.8) \quad \mathcal{R}_{\alpha, N; j}(x', u) := c_\alpha \sum_I (\tilde{\rho}_j(x' - u) - \tilde{\rho}_j(x_I - u)) \chi_I(x').$$

Let T be a bounded linear operator on $L^2(\mathbb{R})$ associated with a scalar kernel $K(x, y)$ defined on $\{(x, y) \in \mathbb{R} \times \mathbb{R} : x \neq y\}$, given initially by $Tf(x) = \int_{\mathbb{R}} K(x, y)f(y)dy$ for $x \notin \text{supp } f$ for $f \in \mathcal{C}_c^\infty(\mathbb{R})$, where $K(x, y)$ satisfies the following conditions ($\mathcal{C}$): there exists a constant $C > 0$ such that for all $x \neq y$,

- (i) $|K(x, y)| \leq C|x - y|^{-1}$;
- (ii) $|K(x, y) - K(x', y)| \leq C|x - x'||x - y|^{-2}$, if $|x - x'| \leq |x - y|/2$;
- (iii) $|K(x, y) - K(x, y')| \leq C|y - y'||x - y|^{-2}$, if $|y - y'| \leq |x - y|/2$;
- (iv) $|K(x, y) - K(x', y) - K(x, y') + K(x', y')| \leq C|x - x'||y - y'||x - y|^{-3}$, if $|x - x'| + |y - y'| \leq |x - y|/2$;
- (v) $T(1) = T^*(1) = 0$.

We denote by $\|K\|_{\mathbb{R}}$ the smallest constant C that satisfies (i)–(iv) above. The operator norm of T is defined by $\|T\|_* := \|T\|_{L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})} + \|K\|_{\mathbb{R}}$.

Proposition A.1. *Suppose that T is an operator as above. Then T is bounded on $L^2(\mathbb{R}, \mathfrak{H})$ and the test function space $\mathcal{M}(\beta, \gamma, r, x_0; \mathfrak{H})$ for $\beta, \gamma \in (0, 1), r > 0$ and $x_0 \in \mathbb{R}$. Moreover, there exists a constant C independent of r and x_0 such that*

$$(A.9) \quad \|T(\mathbf{f})\|_{\square} \leq C\|T\|_* \|\mathbf{f}\|_{\square}.$$

for $\square = L^2(\mathbb{R}, \mathfrak{H})$ or $\mathcal{M}(\beta, \gamma, r, x_0; \mathfrak{H})$.

Proof. The scalar case is [21, Lemma 2.1.9]. (A.9) for $L^2(\mathbb{R}, \mathfrak{H})$ is trivial since T acting on each component of an $L^2(\mathbb{R}, \mathfrak{H})$ -function. Let $A = \|\mathbf{f}\|_{\mathcal{M}(\beta, \gamma, r, x_0; \mathfrak{H})}$. For each $\mathbf{u} \in \mathfrak{H}$ with $\|\mathbf{u}\|_{\mathfrak{H}} = 1$, set $f_{\mathbf{u}}(x) = \langle \mathbf{f}(x), \mathbf{u} \rangle_{\mathfrak{H}}$. Then, by the definition of the $\mathfrak{H}$ -valued $\mathcal{M}$ -norm,

$$\begin{aligned} |f_{\mathbf{u}}(x)| &\leq |\mathbf{f}(x)|_{\mathfrak{H}} \leq A \frac{r^\gamma}{(r + |x - x_0|)^{1+\gamma}}, \\ |f_{\mathbf{u}}(x) - f_{\mathbf{u}}(y)| &= |\langle \mathbf{f}(x) - \mathbf{f}(y), \mathbf{u} \rangle_{\mathfrak{H}}| \leq \|\mathbf{f}(x) - \mathbf{f}(y)\|_{\mathfrak{H}} \\ &\leq A \left(\frac{|x - y|}{r + |x - x_0|} \right)^\beta \frac{r^\gamma}{(r + |x - x_0|)^{1+\gamma}}. \end{aligned}$$

Moreover, the cancellation condition is inherited from $\mathbf{f}$. Hence $\|f_{\mathbf{u}}\|_{\mathcal{M}(\beta, \gamma, r, x_0)} \leq A$. Then,

$$\|Tf_{\mathbf{u}}\|_{\mathcal{M}(\beta, \gamma, r, x_0)} \leq C\|T\|_* \|f_{\mathbf{u}}\|_{\mathcal{M}(\beta, \gamma, r, x_0)} \leq C\|T\|_* A,$$

by the scalar result. In particular,

$$\begin{aligned} |Tf_{\mathbf{u}}(x)| &\leq C\|T\|_* A \frac{r^\gamma}{(r + |x - x_0|)^{1+\gamma}}, \\ |Tf_{\mathbf{u}}(x) - Tf_{\mathbf{u}}(y)| &\leq C\|T\|_* A \left(\frac{|x - y|}{r + |x - x_0|} \right)^\beta \frac{r^\gamma}{(r + |x - x_0|)^{1+\gamma}}. \end{aligned}$$

Since the kernel of T is scalar-valued, we have $\langle T\mathfrak{f}(x), \mathbf{u} \rangle_{\mathfrak{H}} = Tf_{\mathbf{u}}(x)$. Therefore,

$$\begin{aligned} \|T\mathfrak{f}(x)\|_{\mathfrak{H}} &= \sup_{|\mathbf{u}|_{\mathfrak{H}}=1} |\langle T\mathfrak{f}(x), \mathbf{u} \rangle_{\mathfrak{H}}| = \sup_{|\mathbf{u}|_{\mathfrak{H}}=1} |Tf_{\mathbf{u}}(x)| \\ &\leq C\|T\|_* A \frac{r^\gamma}{(r + |x - x_0|)^{1+\gamma}}, \end{aligned}$$

and

$$\begin{aligned} \|T\mathfrak{f}(x) - T\mathfrak{f}(y)\|_{\mathfrak{H}} &= \sup_{|\mathbf{u}|_{\mathfrak{H}}=1} |\langle T\mathfrak{f}(x) - T\mathfrak{f}(y), \mathbf{u} \rangle_{\mathfrak{H}}| = \sup_{|\mathbf{u}|_{\mathfrak{H}}=1} |Tf_{\mathbf{u}}(x) - Tf_{\mathbf{u}}(y)| \\ &\leq C\|T\|_* A \left(\frac{|x - y|}{r + |x - x_0|} \right)^\beta \frac{r^\gamma}{(r + |x - x_0|)^{1+\gamma}}. \end{aligned}$$

The cancellation condition for $T\mathfrak{f}$ follows in the same way, since it holds after taking the inner product with every $\mathbf{u} \in \mathfrak{H}$. Consequently, (A.9) holds. The proof is complete. $\square$

In the decompositions (A.1) and (A.3), the reason for choosing α small and N large is to make the individual error terms small. But this produces more terms than in the usual dyadic decomposition. However, the factor c_α compensates for this loss, as quantified in the following lemma.

Lemma A.1. *Let $r_j = 2^{-\alpha j}$ and $c_\alpha = \alpha \ln 2$. If $\sigma > 0$, then for every $d > 0$,*

$$(A.10) \quad c_\alpha \sum_{j \in \mathbb{Z}} \frac{r_j}{(r_j + d)^{1+\sigma}} \lesssim \frac{1}{d^\sigma},$$

where the implicit constant is independent of $0 < \alpha \leq 1$, but may depend on σ .

Proof. Given $d > 0$, there exists $l \in \mathbb{Z}$ such that $2^{-\alpha(l+1)} \leq d < 2^{-\alpha l}$. Then, we have

$$L.H.S. \leq c_\alpha \sum_{j > l} \frac{2^{-\alpha j}}{d^{1+\sigma}} + c_\alpha \sum_{j \leq l} 2^{\alpha j \sigma} \lesssim \frac{2^{-\alpha l}}{d^{1+\sigma}} + 2^{\alpha l \sigma} \lesssim \frac{1}{d^\sigma}.$$

The lemma is proved. $\square$

Proof of Proposition 5.2. To estimate $\mathcal{R}_{\alpha, N; j}(x', u)$ in (A.8), recall the estimates (3.14) for the Poisson kernel P_t and

$$\rho_t(x) = -t \frac{\partial P_t}{\partial t}(x) = \frac{1}{\pi} \frac{t^2 - x^2}{(t^2 + x^2)^2} t.$$

Set $r_j := 2^{-\alpha j}$. Given $x' \in \mathbb{R}$, let I be the dyadic interval of length $2^{-\alpha(j+N)}$ such that $x' \in I$. Then $|x' - x_I| \leq 2^{-\alpha N} r_j$. By the mean value theorem and definition (5.7), for a nonnegative integer b ,

$$\begin{aligned} (A.11) \quad |\partial_u^b \mathcal{R}_{\alpha, N; j}(x', u)| &= \left| \int_{r_j}^{2^\alpha r_j} \left(\partial_u^b \rho_t(x' - u) - \partial_u^b \rho_t(x_I - u) \right) \frac{dt}{t} \right| \\ &\leq |x' - x_I| \int_{r_j}^{2^\alpha r_j} \sup_{\xi \in I} |\partial_u^{b+1} \rho_t(\xi - u)| \frac{dt}{t} \\ &\lesssim 2^{-\alpha N} r_j \int_{r_j}^{2^\alpha r_j} \sup_{\xi \in I} \frac{t}{(t + |\xi - u|)^{3+b}} \frac{dt}{t} \\ &\lesssim c_\alpha 2^{-\alpha N} \frac{2^\alpha r_j^2}{(r_j + |x' - u|)^{3+b}} \\ &\lesssim c_\alpha 2^{-\alpha N} \frac{r_j}{(r_j + |x' - u|)^{2+b}}, \end{aligned}$$

where we used the estimate for derivatives of the Poisson kernel, and in the last inequality we used $r_j + |\xi - u| \approx r_j + |x' - u|$ for $\xi \in I$ in the second inequality, and $2^\alpha \leq 2$ for $0 < \alpha < 1$ in the last one. Similarly,

$$(A.12) \quad |\partial_x^b \tilde{\varphi}_j(x)| \lesssim \frac{r_j}{(r_j + |x|)^{2+b}},$$

since $\varphi \in \mathcal{S}(\mathbb{R})$. Although $\mathcal{R}_{\alpha,N;j}(x', u)$ is only piecewise continuous, estimates (A.11)-(A.12) allow us to use differentiation under the integral sign (cf. [12, Proposition 2.23]) to show that $\mathcal{R}'_{\alpha,N}$ is smooth and

$$(A.13) \quad \partial_x^a \partial_u^b \mathcal{R}'_{\alpha,N}(x, u) = \sum_j \int_{\mathbb{R}} \partial_x^a \tilde{\varphi}_j(x - x') \partial_u^b \mathcal{R}_{\alpha,N;j}(x', u) dx',$$

for nonnegative integers a, b . Then, by estimates (A.11)-(A.12) and Lemma A.1, we have

$$\begin{aligned} |\partial_x^a \partial_u^b \mathcal{R}'_{\alpha,N}(x, u)| &\lesssim c_\alpha 2^{-\alpha N} \sum_j \int_{\mathbb{R}} \frac{r_j}{(r_j + |x - x'|)^{2+a}} \frac{r_j}{(r_j + |x' - u|)^{2+b}} dx' \\ &\lesssim c_\alpha 2^{-\alpha N} \sum_j \frac{r_j}{(r_j + |x - u|)^{2+a+b}} \\ &\lesssim 2^{-\alpha N} \frac{1}{|x - u|^{1+a+b}}, \end{aligned}$$

where the implicit constants are independent of $0 < \alpha \leq 1$. Consequently, $\mathcal{R}'_{\alpha,N}$ satisfies the Calderón-Zygmund conditions (i)–(iv) in Proposition A.1, with constants bounded by $C2^{-\alpha N}$. Moreover,

$$\int_{\mathbb{R}} \tilde{\varphi}_j(x - x') dx = 0, \quad \int_{\mathbb{R}} \mathcal{R}_{\alpha,N;j}(x', u) du = 0,$$

and hence the cancellation condition also holds.

Similarly, $\mathcal{R}_\alpha$ has the kernel

$$(A.14) \quad \mathcal{R}_\alpha(u) := \sum_{j \in \mathbb{Z}} \int_{r_j}^{2^\alpha r_j} \left(\varphi_t * (\rho_t - \tilde{\rho}_j) + (\varphi_t - \tilde{\varphi}_j) * \tilde{\rho}_j \right)(u) \frac{dt}{t}.$$

For $t \in [r_j, 2^\alpha r_j]$ and a nonnegative integer q , we have the estimate

$$\begin{aligned} |\partial_u^q (\rho_t(u) - \tilde{\rho}_j(u))| &= \left| \frac{1}{c_\alpha} \int_{r_j}^{2^\alpha r_j} \partial_u^q (\rho_t(u) - \rho_{t'}(u)) \frac{dt'}{t'} \right| \\ &= \left| \frac{1}{c_\alpha} \int_{r_j}^{2^\alpha r_j} \int_{t'}^t s \partial_s \partial_u^q \rho_s(u) \frac{ds}{s} \frac{dt'}{t'} \right| \\ &\lesssim \frac{1}{c_\alpha} \int_{r_j}^{2^\alpha r_j} \int_{\min\{t, t'\}}^{\max\{t, t'\}} \frac{s}{(s + |u|)^{2+q}} \frac{ds}{s} \frac{dt'}{t'} \\ &\lesssim \alpha \frac{r_j}{(r_j + |u|)^{2+q}}, \end{aligned}$$

by estimate (3.14) for derivatives of the Poisson kernel P_t , where $0 \leq |\ln \frac{t}{t'}| \leq c_\alpha = \alpha \ln 2$. The same estimates hold for $\varphi_t - \tilde{\varphi}_j$. Together with the corresponding estimates for φ_t and $\tilde{\rho}_j$, the same argument as above shows that $\mathcal{R}_\alpha$ satisfies the Calderón-Zygmund conditions (i)–(iv) in Proposition A.1, with constants bounded by $C\alpha$. The cancellation condition follows from the cancellations of φ_t , $\tilde{\varphi}_j$, ρ_t , and $\tilde{\rho}_j$.

We will use the quantitative form of the $T(1)$ theorem of David–Journé (cf. e.g. [8])

$$\|T\|_{L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})} \lesssim \|K\|_{\text{CZ}} + \|T\|_{\text{WBP}} + \|T(1)\|_{\text{BMO}} + \|T^*(1)\|_{\text{BMO}},$$

where $\|K\|_{\text{CZ}}$ denotes the constants in the standard Calderón–Zygmund kernel estimates and $\|T\|_{\text{WBP}}$ is the weak boundedness constant. We have already shown $\|K_{\mathcal{R}_\alpha}\|_{\text{CZ}} \lesssim \alpha$ for $\mathcal{R}_\alpha$. Write $K_{j,t} := \varphi_t * (\rho_t - \tilde{\rho}_j) + (\varphi_t - \tilde{\varphi}_j) * \tilde{\rho}_j$ with $t \in [r_j, 2^\alpha r_j]$. The estimates above imply

$$\int_{\mathbb{R}} K_{j,t}(u) du = 0, \quad |K_{j,t}(u)| \lesssim \alpha \frac{r_j}{(r_j + |u|)^2}, \quad \|K_{j,t}\|_\infty \lesssim \frac{\alpha}{r_j}.$$

Consequently, $\mathcal{R}_\alpha(1) = 0 = (\mathcal{R}_\alpha)^*(1)$. It remains to verify the weak boundedness property for $\mathcal{R}_\alpha$.

Let Q be an interval with $R = \ell(Q)$, and let a, b be normalized C^1 bump functions adapted to Q , i.e. $\text{supp } a, \text{supp } b \subset Q$, $\|a\|_\infty + \|b\|_\infty \lesssim 1$, and $\|a'\|_\infty + \|b'\|_\infty \lesssim R^{-1}$. Using the cancellation of $K_{j,t}$, we obtain

$$\begin{cases} |\langle K_{j,t} * a, b \rangle| \lesssim \alpha r_j \left(1 + \log \frac{R}{r_j}\right), & \text{if } r_j \leq R, \\ |\langle K_{j,t} * a, b \rangle| \lesssim \|K_{j,t}\|_\infty \|a\|_1 \|b\|_1 \lesssim \alpha \frac{R^2}{r_j}, & \text{if } r_j > R. \end{cases}$$

Therefore, since $c_\alpha = \alpha \log 2$ and $r_j = 2^{-\alpha j}$,

$$|\langle \mathcal{R}_\alpha a, b \rangle| \lesssim \alpha c_\alpha \left[\sum_{r_j \leq R} r_j \left(1 + \log \frac{R}{r_j}\right) + \sum_{r_j > R} \frac{R^2}{r_j} \right] \lesssim \alpha R.$$

Hence $\|\mathcal{R}_\alpha\|_{\text{WBP}} \lesssim \alpha$. Together with $\|K_{\mathcal{R}_\alpha}\|_{\text{CZ}} \lesssim \alpha$, $\mathcal{R}_\alpha(1) = \mathcal{R}_\alpha^*(1) = 0$, the quantitative $T(1)$ theorem yields $\|\mathcal{R}_\alpha\|_{L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})} \lesssim \alpha$. Thus, we can apply Proposition A.1 to obtain that

$$(A.15) \quad \|\mathcal{R}_\alpha\|_{\square} \lesssim \alpha \lesssim \frac{1}{3},$$

provided we choose α sufficiently small, where $\|\cdot\|_{\square}$ is the operator norm on the space of linear endomorphisms of $\square = L^2(\mathbb{R}, \mathfrak{H})$ or $\mathcal{M}(\beta, \gamma, r, x_0; \mathfrak{H})$.

For the operator $\mathcal{R}'_{\alpha,N}$, we have already shown $\|K_{\mathcal{R}'_{\alpha,N}}\|_{\text{CZ}} \lesssim 2^{-\alpha N}$, and $\mathcal{R}'_{\alpha,N}(1) = 0 = (\mathcal{R}'_{\alpha,N})^*(1)$ by the cancellation identities established above. As above, we can verify $\|\mathcal{R}'_{\alpha,N}\|_{\text{WBP}} \lesssim 2^{-\alpha N}$. Thus, $\|\mathcal{R}'_{\alpha,N}\|_{L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})} \lesssim 2^{-\alpha N}$. So we can apply Proposition A.1 again to get

$$(A.16) \quad \|\mathcal{R}'_{\alpha,N}\|_{\square} \lesssim 2^{-\alpha N} \leq \frac{1}{3},$$

if we choose N sufficiently large. Therefore,

$$(A.17) \quad \mathcal{T} = id - \mathcal{R}'_{\alpha,N} - \mathcal{R}_\alpha$$

is invertible and its inverse $\mathcal{T}^{-1}$ is also bounded on $\square$, with a bound independent of r, x_0 . Consequently, for $\tilde{\phi}_j(\cdot, x_I)$ given by (A.6), $\mathcal{T}^{-1}(\tilde{\phi}_j(\cdot, x_I))$ is also in $\mathcal{M}(\beta, \gamma, 2^{-\alpha j}, x_I; \mathfrak{H})$. Denote it by $\phi_j(x, x_I)$. Then applying $\mathcal{T}^{-1}$ to (A.5), we get

$$(A.18) \quad \mathfrak{f}(x) = \mathcal{T}^{-1} \mathcal{T} \mathfrak{f} = \sum_j \sum_I c_\alpha \ell(I) |\tilde{\rho}_j * \mathfrak{f}(x_I)|_{\mathfrak{H}} \phi_j(x, x_I),$$

in $L^2(\mathbb{R}, \mathfrak{H})$. Here $\|\phi_j(\cdot, x_I)\|_{\mathcal{M}(\beta, \gamma, 2^{-\alpha j}, x_I; \mathfrak{H})}$ are uniformly bounded by Proposition A.1. $\square$

Proof of Proposition 5.3. Convolving the wavelet Calderón reproducing formula (5.6) with ψ_t we obtain for each choice of $x_I \in I$,

$$(A.19) \quad \psi_t * \mathbf{f}(v) = \sum_{j \in \mathbb{Z}} \sum_I c_\alpha \ell(I) |\tilde{\rho}_j * \mathbf{f}(x_I)|_{\mathfrak{H}} (\psi_t * \phi_j(\cdot, x_I))(v).$$

where $\ell(I) = 2^{-\alpha(j+N)}$ and $\|\phi_j(\cdot, x_I)\|_{\mathcal{M}(\beta, \gamma, 2^{-\alpha j}, x_I; \mathfrak{H})} \leq C$. We can choose $x_I \in I$ so that

$$(A.20) \quad |\tilde{\rho}_j * \mathbf{f}(x_I)|_{\mathfrak{H}} = \inf_{x \in I} |\tilde{\rho}_j * \mathbf{f}(x)|_{\mathfrak{H}},$$

by the continuity of $\tilde{\rho}_j * \mathbf{f}$. But for $2^{-\alpha k} \leq t < 2^{-\alpha(k-1)}$, we have the following almost orthogonality estimate for $\phi_j(\cdot, x_I)$ and ψ_t (cf. [23, Lemma 6]),

$$(A.21) \quad |(\psi_t * \phi_j(\cdot, x_I))(v)|_{\mathfrak{H}} \leq C 2^{-\alpha|k-j|\beta} \frac{r_{j,k}^\gamma}{(r_{j,k} + |v - x_I|)^{1+\gamma}}, \quad r_{j,k} := 2^{-\alpha(j \wedge k)}.$$

For fixed $v \in C_0 J$ with $\ell(J) = 2^{-\alpha(k+N)}$, decompose the intervals I according to their distance from v :

$$\begin{aligned} \mathfrak{T}_0 &= \{I : \ell(I) = 2^{-\alpha(j+N)}, |v - x_I| \leq r_{j,k}\}, \\ \mathfrak{T}_l &= \{I : \ell(I) = 2^{-\alpha(j+N)}, 2^{\alpha(l-1)} r_{j,k} \leq |v - x_I| \leq 2^{\alpha l} r_{j,k}\}, \quad l = 1, 2, \dots \end{aligned}$$

Then, for $I \in \mathfrak{T}_l$,

$$\frac{r_{j,k}^\gamma}{(r_{j,k} + |v - x_I|)^{1+\gamma}} \lesssim 2^{\alpha(j \wedge k)} 2^{-\alpha l(1+\gamma)}.$$

Moreover, $\bigcup_{I \in \mathfrak{T}_l} I$ and J are contained in an interval $\tilde{I}$ with length $|\tilde{I}| \lesssim r_{j,k} 2^{\alpha l}$, by $\ell(J) = 2^{-\alpha(k+N)} \leq r_{j,k}$. Let

$$\max \left\{ \frac{1}{1+\gamma}, \frac{1}{1+\beta} \right\} < r < 1 \quad \text{and} \quad F_j(x) := \sum_{\ell(I)=2^{-\alpha(j+N)}} |\tilde{\rho}_j * \mathbf{f}(x_I)|_{\mathfrak{H}}^r \chi_I(x),$$

we have

$$\begin{aligned} (A.22) \quad \sum_{I \in \mathfrak{T}_l} \ell(I) |\tilde{\rho}_j * \mathbf{f}(x_I)|_{\mathfrak{H}} &\leq \ell(I)^{1-\frac{1}{r}} \left(\sum_{I \in \mathfrak{T}_l} \ell(I) |\tilde{\rho}_j * \mathbf{f}(x_I)|_{\mathfrak{H}}^r \right)^{\frac{1}{r}} \\ &\lesssim \ell(I)^{1-\frac{1}{r}} \left(\int_{\tilde{I}} F_j(x) dx \right)^{\frac{1}{r}} \\ &\lesssim 2^{-\alpha j(1-\frac{1}{r})} (r_{j,k} 2^{\alpha l})^{\frac{1}{r}} M(F_j)(u)^{\frac{1}{r}}, \end{aligned}$$

where $u \in J$, and the implicit constant may depend on C_0 . Substituting (A.21)-(A.22) into (A.19) and taking summation over l , we obtain

$$(A.23) \quad \sup_{v \in C_0 J} |\psi_t * \mathbf{f}(v)|_{\mathfrak{H}} \chi_J(u) \lesssim \sum_j A_{k,j} B_j(u) \chi_J(u),$$

where $A_{k,j} := 2^{-\alpha|k-j|\beta} 2^{\alpha[(j \wedge k) - j](1-\frac{1}{r})}$, $B_j(u) := M(F_j)(u)^{1/r}$. Consequently,

$$\begin{aligned} \sum_k \sum_j \int_{r_k}^{2^\alpha r_k} \sup_{v \in C_0 J} |\psi_t * \mathbf{f}(v)|_{\mathfrak{H}}^2 \chi_J(u) \frac{dt}{t} &\lesssim \sum_k \left(\sum_j A_{k,j} B_j(u) \right)^2 \\ &\lesssim \sum_j B_j(u)^2 = \sum_j M(F_j)(u)^{2/r}, \end{aligned}$$

Here the second inequality follows from the discrete Young inequality on $\ell^2(\mathbb{Z})$. Indeed, since $r > 1/(1+\beta)$, there exists $\delta > 0$ such that $A_{k,j} \lesssim 2^{-\alpha\delta|k-j|}$. Now taking square roots and L^1 -norms, and applying the Fefferman-Stein vector-valued maximal inequality with $p = 1/r > 1$, we get

$$\begin{aligned} \left\| \left(\sum_{k,j} \int_{r_k}^{2^\alpha r_k} \sup_{v \in C_0 J} |\psi_t * \mathfrak{f}(v)|_{\mathfrak{H}}^2 \chi_J \frac{dt}{t} \right)^{1/2} \right\|_{L^1} &\lesssim \left\| \left(\sum_j M(F_j)^{2/r} \right)^{1/2} \right\|_{L^1} \\ &= \left\| \left(\sum_j M(F_j)^{2/r} \right)^{r/2} \right\|_{L^{1/r}}^{1/r} \lesssim \left\| \left(\sum_j F_j^{2/r} \right)^{r/2} \right\|_{L^{1/r}}^{1/r}. \end{aligned}$$

Since the intervals I with fixed j form an almost disjoint covering of $\mathbb{R}$ (only endpoints may overlap), we have

$$F_j^{2/r} = \sum_{\ell(I)=2^{-\alpha(j+N)}} |\tilde{\rho}_j * \mathfrak{f}(x_I)|_{\mathfrak{H}}^2 \chi_I \quad a.e.$$

By the choice of x_I in (A.20),

$$\begin{aligned} |\tilde{\rho}_j * \mathfrak{f}(x_I)|_{\mathfrak{H}}^2 &\lesssim \inf_{x \in I} |\tilde{\rho}_j * \mathfrak{f}(x)|_{\mathfrak{H}}^2 = \frac{1}{c_\alpha^2} \inf_{x \in I} \left| \int_{r_j}^{2^\alpha r_j} \rho_t * \mathfrak{f}(x) \frac{dt}{t} \right|_{\mathfrak{H}}^2 \\ &\leq \frac{1}{c_\alpha^2} \inf_{x \in I} \left\{ \int_{r_j}^{2^\alpha r_j} |\rho_t * \mathfrak{f}(x)|_{\mathfrak{H}}^2 \frac{dt}{t} \right\} \leq \frac{1}{c_\alpha} \inf_{x \in I} \int_{r_j}^{2^\alpha r_j} |\rho_t * \mathfrak{f}(x)|_{\mathfrak{H}}^2 \frac{dt}{t}, \end{aligned}$$

using the Cauchy–Schwarz inequality. The desired Plancherel–Pólya inequality follows. $\square$

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